MATH 56 WORKSHEET : Aliasing error

Barnett 4/22/14.

Given a 2π-periodic func f, how good an approx is $\tilde{f}_m$ to $\hat{f}_m$?

recall:
- DFT $\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j$ ← where sample $f_j = \frac{1}{N} f\left(\frac{2\pi j}{N}\right)$
- Fourier series $f(x) = \sum_{m\in\mathbb{Z}} \hat{f}_m e^{imx}$

A) Take the DFT of samples of f written as a Fourier series:

ie $\tilde{f}_m$ = "DFT (Fourier series for f), m^{th} component thereof"

= . . .

[did you pick a sum index not clashing with m ?]

Swap the order of Σ's & use a lemma to simplify:

Write in following form: fill in the missing frequency indices:

$$\tilde{f}_m = \underbrace{\cdots + \hat{f}_{\bigcirc}}_{\text{'aliasing' error}} + \underbrace{\hat{f}_m}_{\text{what you want}} + \underbrace{\hat{f}_{\bigcirc} + \hat{f}_{\bigcirc} + \cdots}_{\text{'aliasing' error}}$$

B) If $\hat{f}_n = 0$ for $|n| \geq N/2$, what is aliasing error in $\tilde{f}_m$, $m < N/2$?

BONUS: if $|\hat{f}_n| \leq C r^{|n|}$, ie exponential decay, find a bound on aliasing error (is it small?)

~ SOLUTIONS ~

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<u>Given a 2π-periodic func f, how good an approx is $\tilde{f}_m$ to $\hat{f}_m$?</u>

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- DFT $\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j$ ← where sample $f_j = \frac{1}{N} f\left(\frac{2\pi j}{N}\right)$
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A) Take the DFT of samples of f written as a Fourier series:

ie $\tilde{f}_m$ = "DFT (Fourier series for f), m^{th} component thereof"

$$= \sum_{j=0}^{N-1} \omega^{-mj} \frac{1}{N} \sum_{n\in\mathbb{Z}} \hat{f}_n \underbrace{e^{in\boxed{\frac{2\pi j}{N}}}}_{\left(e^{\frac{2\pi i}{N}}\right)^{nj} = \omega^{nj}}$$

x_j, the j^{th} node.

[did you pick a sum index not clashing with m ?] yes, I picked n.

Swap the order of Σ's & use a lemma to simplify:

$$= \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{1}{N} \sum_{j=0}^{N-1} \underbrace{\omega^{-mj}\omega^{nj}}_{\omega^{(n-m)j}}$$

sum lemma $= \begin{cases} N & n\equiv m \pmod N \\ 0 & \text{otherwise} \end{cases}$

$$= \sum_{\substack{n\in\mathbb{Z} \\ n\equiv m \pmod N}} \hat{f}_n$$

Write in following form: fill in the missing frequency indices:

$$\tilde{f}_m = \underbrace{\cdots + \hat{f}_{m-N}}_{\text{'aliasing' error}} + \underbrace{\hat{f}_m}_{\text{what you want}} + \underbrace{\hat{f}_{m+N} + \hat{f}_{m+2N} \cdots}_{\text{'aliasing' error}}$$

B) If $\hat{f}_n = 0$ for $|n| \geq N/2$, what is aliasing error in $\tilde{f}_m$, $m < N/2$?

Error is zero for each such m, since $|m+kN| \geq N/2$ for $k\neq 0$.

BONUS: if $|\hat{f}_n| \leq C r^{|n|}$, ie exponential decay, find a bound on aliasing error (is it small?)

aliasing error $\left|\tilde{f}_m - \hat{f}_m\right| \leq \sum_k |\hat{f}_{m+kN}| + |\hat{f}_{m-kN}| \leq C\sum r^{m+kN} + r^{m-kN} = O\left(r^{N/2}\right)$ by geom series.