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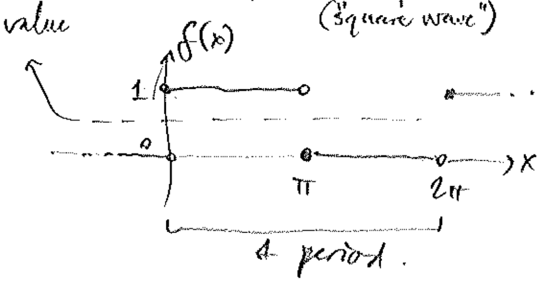
5/8/13

SOLUTIONS

Math 56 Compu & Expt Math, Spring 2013: Quiz 2

in X-hr 5/8/13, 35 mins, just pencil and paper

sketch periodized function: ("square wave")



1. (a) Compute the Fourier series coefficients $\hat{f}_m$ for [4]

$$f(x) = \begin{cases} 1, & 0 \le x < \pi, \\ 0, & \pi \le x < 2\pi. \end{cases}$$

Your answer shouldn't involve any exponentials.

projection formula:

$$\hat{f}_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx}\, dx$$

using support of f to reduce domain

$$= \frac{1}{2\pi}\int_0^{\pi} e^{-imx}\, dx \overset{m\neq 0}{=} \frac{1}{-2\pi i m}\underbrace{\left[e^{-imx}\right]_0^{\pi}}$$

$1 - 1 = 0$, m even

$-1 - 1 = -2$, m odd

$$= \begin{cases} 1/2, & m = 0 \\ 0, & m \text{ even } \neq 0 \\ \dfrac{1}{i\pi m} & m \text{ odd.} \end{cases}$$

[2] (b) Compute the sum of the squared magnitudes of the Fourier coefficients [Hint: don't use (a)].

I was lenient about this in grading.

Parseval says $\|f\|_2^2 = 2\pi \sum_{m\in\mathbb{Z}} |\hat{f}_m|^2$

so $$\sum_{m\in\mathbb{Z}} |\hat{f}_m|^2 = \frac{1}{2\pi}\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \frac{1}{2\pi}\int_0^{\pi} 1^2\, dx = \frac{1}{2}$$

Using (a) is possible but only if you realize $\sum_{m \text{ odd}} \frac{1}{m^2} = \sum_{m=1}^{\infty}\frac{1}{m^2} - \sum_{m=1}^{\infty}\frac{1}{(2m)^2}$

[1] (c) Is it possible that there is a set of complex numbers $\{d_m\}_{|m|<10}$ such that $\|\sum_{|m|<10} d_m e^{imx} - f(x)\|$ is smaller than $\|\sum_{|m|<10} \hat{f}_m e^{imx} - f(x)\|$?

in $L^2(0,2\pi)$

Nope, not possible since $\hat{f}_m$ are the best-approximating set of coeffs d_m, for any fixed set of indices, eg. $|m|<10$ as here.

Fou. coeffs all zero except $\hat{f}_7 = 1$.

Fourier series used will be $|m| < N/2 = 2$ ie $m = \{-1, 0, 1\}$ freq's.

[3] 2. What interpolant function is produced if $N = 4$ point trigonometric polynomial interpolation is carried out on the function e^{i7x}, on the usual interval $(0, 2\pi)$?

means: take DFT of samples f_j & interpret as $N/2$-truncated Fourier series.

Aliasing formula

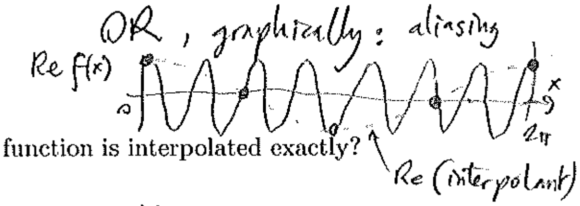
$$\tilde{f}_m = \cdots + \hat{f}_{m-N} + \hat{f}_m + \hat{f}_{m+N} + \underbrace{\hat{f}_{m+2N}}_{} + \cdots$$

$m=-1$ gives $\hat{f}_7$ here, which $= 1$

so $\tilde{f}_{-1} = 1$, other zero.

$$\Rightarrow \sum_{|m|<2} \tilde{f}_m e^{imx} \text{ gives } e^{-ix} \quad \text{interpolant.}$$

OR, graphically: aliasing



[BONUS] How many sample points are needed so that the previous function is interpolated exactly?

Nyquist Thm says $|m| < N/2$ interpolated exactly, & $m = 7$, so $N > 14$ ie $N = 15, 16$, etc. will do.

[4] 3. A function f has Fourier coefficients bounded by $\hat{f}_m \le r^{+|m|}$ for some $r < 1$. As $N \to \infty$, what type of error convergence do you expect when $\hat{f}_m$ is approximated by $\tilde{f}_m$, i.e. via the DFT of N samples?

Intuitively, coeffs decay exponentially, so expect exponential convergence of error $|\tilde{f}_m - \hat{f}_m|$. Let's prove it:

Derive a *rigorous* upper bound on this error (you may assume N is sufficiently big):

Aliasing $$\tilde{f}_m = \cdots + \hat{f}_{m-N} + \hat{f}_m + \hat{f}_{m+N} + \cdots$$

$$\text{so } |\tilde{f}_m - \hat{f}_m| = \left| \sum_{k=1}^{\infty} \hat{f}_{m-kN} + \sum_{k=1}^{\infty} \hat{f}_{m+kN} \right|$$

$$\le \sum_{k=1}^{\infty} |\hat{f}_{m-kN}| + \sum_{k=1}^{\infty} |\hat{f}_{m+kN}|$$

$$\le \sum_{k=1}^{\infty} r^{|m-kN|} + \sum_{k=1}^{\infty} r^{|m+kN|}$$

now assume $N \ge |m|$, so $|m - kN| = kN - m$

$$= r^{-m} \sum_{k=1}^{\infty} r^{kN} + r^{m} \sum_{k=1}^{\infty} r^{kN}$$

$$= (r^{-m} + r^m)\, r^N \underbrace{\sum_{k=0}^{\infty} r^{kN}}_{}$$

geom. series $= \dfrac{1}{1 - r^N}$

$$= \underbrace{\frac{r^{-m} + r^m}{1 - r^N}}_{C_m} \cdot r^N = O(r^N)$$

indeed exponentially convergent

[3] 4. Compute the result when the vector $\underset{\vec{f}}{[1\,2\,3]}$ is acyclically convolved with the vector $\underset{\vec{g}}{[3\,2\,1]}$.

you know answer has length N_1+N_2-1 $=5$.

$$(\vec{f} *_A \vec{g})_j = \sum_{i\in\mathbb{Z}} f_i\, g_{j-i} \qquad \text{acyclic.}$$

Easier is to sum multiples of f shifted by each integer:

$$\begin{array}{ccccc} 1 & 2 & 3 & & \\ & 1 & 2 & 3 & \\ & & 1 & 2 & 3 \end{array} \quad \xrightarrow{\text{now mult by entries of } g_i} \quad \begin{array}{ccccc} 3 & 6 & 9 & & \\ & 2 & 4 & 6 & \\ + & & 1 & 2 & 3 \\ \hline [3 & 8 & 14 & 8 & 3] \end{array} \quad \begin{array}{l} \leftarrow 3\times f \\ \leftarrow 2\times f \text{ shifted } 1 \\ \leftarrow f \text{ shifted } 2. \\ \leftarrow f*g. \end{array}$$

[3] 5. Roughly how many times faster would you expect Strassen's algorithm to multiply two numbers of length 10^6 digits to run than the standard long multiplication algorithm? (Explain.)

Strassen does convolution by FFTs (length $2N-1$), needs 3 of them. ie $O(6N \log_2 2N)$ flops

Naive is $O(2N^2)$ flops. $\longrightarrow$ ratio roughly $\dfrac{N}{3\log_2 N} \approx \dfrac{10^6}{60}$

$\approx 1.6\times 10^4$.

However, really the dominant thing is it's nearly $O(N)$ faster, so an answer of 10^6 is not too far off.

20 pts total.