MATH 56 WORKSHEET : Stability & error analysis

4/9/13
Barnett

A) Is machine arithmetic for $f(x) = 1 + x$ backwards stable?
[Hint: You only have <u>one</u> input datum x ; check it for all values of x]

B) [Review] What is κ (relative condition number) for the above problem?
Does it blowing up happen at the same x as the issue in A)?

Bonus C) Show that any route via the characteristic polynomial for eigenvalues cannot be backwards stable:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⟵ write its char poly $a_2\lambda^2 + a_1\lambda + a_0 = 0$

Find the eigenvalues for the v. close poly, $\lambda^2 - 2\lambda + 1 - 10^{-16} = 0$

How far from the original eigenvalues are they?

Are there any $O(\varepsilon_{mach})$ perturbations of A with these eigenvalues?

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SOLUTIONS

A) Is machine arithmetic for $f(x) = 1 + x$ backwards stable?
[Hint: You only have one input datum x ; check it for all values of x]

machine does

$$\tilde{f}(x) = \underbrace{fl(1)}_{\text{all this 1 exactly.}} \oplus fl(x)$$

$$= [1 + (1+\varepsilon_1)x](1+\varepsilon_2)$$

$$= \cancel{1} + \cancel{x} + \varepsilon_2 + x(\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2) \xleftrightarrow{\text{equate these}} = \cancel{1} + \cancel{x} + x\varepsilon$$

defn. of bkw stability

$$\tilde{f}(x) = f(x(1+\varepsilon)) \quad \text{for some } \varepsilon.$$

$$= 1 + x(1+\varepsilon)$$

solve for ε: so $\varepsilon = \underbrace{\frac{\varepsilon_2}{x}}_{\text{can get arbitrarily large for } x \to 0.} + \underbrace{\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2}_{O(\varepsilon_{mach})}$

B) [Review] What is κ (relative condition number) for the above problem?
Does it blowing up happen at the same x as the issue in A)?

$$\kappa(x) = \left|\frac{f'(x)x}{f(x)}\right| = \left|\frac{1\cdot x}{1+x}\right| = \left|\frac{x}{1+x}\right| \to \infty \text{ only as } x \to -1.$$

$\kappa = O(1)$ for $x \to 0$, so this is a different issue than A.

Bonus C) Show that any route via the characteristic polynomial for eigenvalues cannot be backwards stable:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow \text{ write its char poly } a_2\lambda^2 + a_1\lambda + a_0 = 0$$

$$(1-\lambda)^2 - 0^2 = \underbrace{\lambda^2}_{a_2=1} - \underbrace{2\lambda}_{a_1=-2} + \underbrace{1}_{a_0=1} = 0$$

Find the eigenvalues for the v. close poly, $\lambda^2 - 2\lambda + \overbrace{1 - 10^{-16}}^{a_0 \text{ tweaked by } \varepsilon_{mach}} = 0$

quadr. formula exactly gives $\lambda = \frac{1}{2}\left(2 \pm \sqrt{4 - 4 + 4\cdot 10^{-16}}\right) = 1 \pm 10^{-8}$

How far from the original eigenvalues are they? 10^{-8} , ie 8 digits worse than ε_{mach}.

Are there any $O(\varepsilon_{mach})$ perturbation of A with these eigenvalues? No

requires perturbation theory of eigenvalues. (try it)