Math 56 Compu & Expt Math, Spring 2013: Midterm 2

5/14/13, pencil and paper, 2 hrs, 50 points. Show working. Good luck!

1. [8 points] Consider $f(x)$ a 2π-periodic bounded function with Fourier coefficients $\hat{f}_m$.

   (a) Assuming $f(x)$ is real-valued, prove that $\hat{f}_{-m} = (\hat{f}_m)^*$ holds for any integer m.

   (b) Derive the kth Fourier coefficient of the function $[f(x)]^2$, in terms of the coefficients $\hat{f}_m$.

(c) Recognize your previous result as an operation (which one?) applied to the discrete set $\{\hat{f}_m\}_{m\in\mathbb{Z}}$ resulting in the set of Fourier coefficients of f^2.

BONUS If f is even, $f(-x) = f(x)$ for all x, what is the consequence for the Fourier coefficients?

2. [10 points]

   (a) Compute the Fourier coefficients of the 2π-periodic function defined by $f(x) = |x|$ in $(-\pi, \pi)$. [Hint: a sketch may help]

(b) Derive a useful bound on the maximum error of approximating the above function f using N-point trigonometric polynomial interpolation, and state its type and order/rate. [Hint: you should get error vanishing as $N \to \infty$. If you cannot, recheck (a).]

(c) Now say trigonometric polynomial interpolation with $N = 8$ points is performed on the function $f(x) = e^{-3ix}$. Give the vector resulting from the discrete Fourier transform (DFT) of the sample vector:

Finally, what interpolant function is produced, and what it its $L^2(0, 2\pi)$ error?

3. [6 points] Consider the 2π-periodic function $f(x) = |\sin^5 x|$ from homework, which is C^4 continuous.

   (a) What can you say about the decay of its Fourier coefficients? (You may state a result without proof.)

   (b) Find a bound on the absolute *error* in the zeroth Fourier coefficient due to approximating it by the zeroth component of the DFT of f sampled on a regular N-point grid.

   [BONUS] Show that (b) gives a bound on the error of a quadrature scheme for $\int_0^{2\pi} f(x)dx$.

4. [10 points]

   (a) Find the $N = 4$ periodic convolution of $[1\,2\,3\,0]$ and $[0\,1\,1\,1]$.

(b) Let $N > 0$ be even. What is the DFT of the length-N vector $[1, -1, 1, -1, \dots, -1]$?

(c) Recall that the DFT is defined by $\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j$ where ω is the principal Nth root of 1. State and *prove* the inversion formula that recovers f_j in terms of $\tilde{f}_m$:

(d) It is easier in practice to deconvolve a signal (or image) that has been blurred by a smooth aperture function or by a discontinuous one? Explain.

5. [6 points]

(a) Say you want to build an arbitrary-precision reciprocal, that given z to N-digit relative accuracy, can compute $1/z$ to the same relative accuracy. Explain how do it (you may make use of other known algorithms) in the minimum complexity (with respect to N) you can.

(b) What complexity is your scheme?

6. [10 points] Short unrelated questions.

(a) Give the precise definition that a function $f(n)$ has super-algebraic convergence to zero as $n \to \infty$.

(b) Up to what power of x do you need to include in the Taylor expansion to $\tan^{-1} x$ to achieve 1000000 digits accuracy at $x = 1/3$? (show working)

(c) Roughly how many Brent–Salamin iterations do you need to approximate π to 1000000 digits accuracy? (show working)

(d) Filtering. You record a signal vector of length 10^6 of audio sampled at a rate of 10^4 per second. By mistake noise ("hum") at the single frequency of 60 Hz corrupted the recording (this is common). Which mode index/indices should you set to zero in the vector's DFT to remove this noise?