Math 54: Topology

Syllabus, problems and solutions

Pierre Clare

Dartmouth College, Summer 2015

Textbook

[**M**] *Topology (2nd ed.)*, by J. Munkres.

Effective syllabus

O. Introduction - *4 lectures*

O.1 Elementary set theory
O.2 Basics on metric spaces

I. Topological spaces and continuous functions - *16 lectures*

I.1. Topologies
I.2. Bases and subbases
I.3. Closed sets
I.4. Continuous maps and the category **Top**.
I.5. Topologies on cartesian products
I.6. Metrizable topologies

II. Connectedness - *2 lectures*

II.1. Connected spaces
II.2. Path connectedness
II.3. Connected components

III. Compactness - *5 lectures*

III.1. Compact spaces
III.2. Fréchet and sequential compactness
III.3. Local compactness, Alexandrov Compactification

IV. Other topics - *3 lectures*

IV.1. Separation axioms
IV.2. The Urysohn Lemma and the Urysohn Metrization Theorem
IV.3. Normed linear spaces
IV.4. Topological properties of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$

Contents

Week 1

Lecture 1. General introduction. (Why) should we study Topology? What is it?
Poincaré (1895): *Analysis situs.*

> There is a science called *analysis situs* and **which has for its object the study of the positional relations of the different elements of a figure, apart from their sizes. This geometry is purely qualitative; its theorems would remain true if the figures, instead of being exact, were roughly imitated by a child**. [...] The importance of *analysis situs* is enormous and can not be too much emphasized [...].
>
> H. Poincaré, *The value of Science*, 1905.

Intuitive notions of neighborhood and deformation.
Fun: example of topological problem: the seven bridges of Königsberg.

Week 2

Lecture 2. [**M**, §6-7]
Finite sets and cardinality. Examples and comparison of infinite subsets of $\mathbb{Z}_+$ (evens, odds, squares, primes). Bijection between two line segments.
Countably infinite and countable sets. Examples: $\mathbb{Z}$ and $\mathbb{Z}_+ \times \mathbb{Z}_+$ are countably infinite.
Fun: the set $S = \{n^2\ ,\ n \in \mathbb{Z}_+\}$ is in bijection with $\mathbb{Z}$. However, there are 'more' integers than squares in the sense that $\sum\limits_{k\in\mathbb{Z}_+} \frac{1}{k} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges while $\sum\limits_{k\in S} \frac{1}{k} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.
What about $\sum\limits_{p\,\text{prime}} \frac{1}{p}$?

Lecture 3. [**M**, §7]
Characterization of countable sets: B is countable iff $\mathbb{Z}_+ \xrightarrow{\text{surj.}} B$ iff $B \xrightarrow{\text{inj.}} \mathbb{Z}_+$.
Infinite subset of $\mathbb{Z}_+$ are countable. Examples of countable sets:

- $\mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbb{Q}$,...
- subsets, *countable* unions, *finite* products of countable sets.

There exist uncountable sets: $\{0,1\}^\omega$ (diagonal extraction argument) and $\mathbb{R}$ for instance.

Week 3

Lecture 4. Notion of distance, metrics. Examples of metric spaces: $(\mathbb{R}^2, \text{Euclidean})$, $(\mathbb{R}^2, \text{Manhattan})$, $\mathbb{Z}$ with the ordinary metric, $(\mathbb{Z}, 2\text{-adic})$. Balls in metric spaces.
Continuous functions on $\mathbb{R}$: definition and expression in terms of balls and inverse images.
Fun: p-adic metric on $\mathbb{Q}$.

Lecture 5. Open sets in metric spaces. Examples in $(\mathbb{R}, \text{Euclidean})$.
Theorem [MT]: let (E, d) be a metric space. Then,

- E and $\varnothing$ are open;
- arbitrary unions of open sets are open;
- finite intersections of open sets are open.

Continuous maps between metric spaces: definition in terms of balls.
Theorem [MC]: a map between metric spaces is continuous if and only if the inverse image of any open set is an open set.

Lecture 6. [**M**, §12-13]
Topology on a set. Discrete and trivial topologies. Finite complement topology.
Reformulation of Theorem [MT]: open sets defined by a metric constitute a topology.
Comparison between topologies: notion of finer (stronger) and coarser (weaker) topology.
Basis for a topology. Examples: disks and rectangles in $\mathbb{R}^2$.

Lecture 7. [**M**, §13]
Topology $\mathcal{T}(\mathcal{B})$ generated by a basis $\mathcal{B}$. Basis elements are open ($\mathcal{B} \subset \mathcal{T}(\mathcal{B})$).
Description of $\mathcal{T}(\mathcal{B})$ (Lemma 13.1): the open sets of $\mathcal{T}(\mathcal{B})$ are the unions of elements of $\mathcal{B}$.
Criterion to find a basis of a given topology $\mathcal{T}$ on a set X (Lemma 13.2):
if a subset $\mathcal{C}$ of $\mathcal{T}$ is a finer covering[1] of X, then $\mathcal{C}$ is a basis and generates $\mathcal{T}$, i.e. $\mathcal{T}(\mathcal{C}) = \mathcal{T}$.
Topologies can be compared by comparing bases (Lemma 13.1):
$\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$ if and only if $\mathcal{B}'$ is a finer covering of X than $\mathcal{B}$.

Week 4

Lecture 8. [**M**, §13-14]
Topology generated by a subbasis.
Order topology: definition, examples: $\mathbb{R}^2$ and $\{1, 2\} \times \mathbb{Z}_+$ with the lexicographical order.
Comparison with the Euclidean and the discrete topology respectively.

Lecture 9. [**M**, §15]
The product topology: definition, bases. Projections and cylinders.

Lecture 10. [**M**, §16]
The subspace topology: definition, bases. Restriction commutes to products.
The topology of the restricted order may differ from the restricted order topology: cases of $X = \mathbb{R}$ and $Y_1 = [0, 1]$, $Y_2 = [0, 1) \cup \{2\}$. They coincide in the case of a convex subset.

Lecture 11. [**M**, §17]
Closed sets: definition, examples. Properties: stability under arbitrary intersections and finite unions. A topology can be defined by its closed sets. Closed sets in the subspace topology. Closure and interior of a set, closure in the subspace topology.
A topology can be defined by its closure operation:
Theorem [Closure] Let X be a set and $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ a map such that,

- $\gamma(\varnothing) = \varnothing$;
- $A \subset \gamma(A)$;
- $\gamma(\gamma(A)) = \gamma(A)$;
- $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$.

Then the family $\{X \setminus \gamma(A)\,,\ A \in \mathcal{P}(X)\}$ is a topology in which $\overline{A} = \gamma(A)$.

[1] If $\mathcal{E}$ and $\mathcal{F}$ are families of subsets of X, we say that $\mathcal{F}$ is a *finer covering of X than $\mathcal{E}$* if for every '$x \in E \in \mathcal{E}$' there exists $F \in \mathcal{F}$ such that $x \in F \subset E$

Week 5

Lecture 12. [**M**, §17]
Characterization of the closure and accumulation points.

Lecture 13. [**M**, §17]
Theorem [MUL] In a metric space, if a sequence converges, it has a unique limit.
Convergent sequences in topological spaces. Uniqueness of limits in Hausdorff (T_2) spaces.
Characterization of accumulation points in T_1 spaces.

Midterm 1. Divisor topology on $\mathbb{Z}_+$. Equivalent metrics generate the same topology. Topology generated by the union of two topologies. Examples of subspaces of $\mathbb{R}$, $[-1, 1]$ and $\mathbb{R}_\ell \times \mathbb{R}_u$.

Post-exam fun: *Sperner's Lemma and consequences*, by J. Bass.

Guest lecture. *The problem with topology*, by James Binkoski (Dartmouth College), an introduction to T. Maudlin's *New foundations for physical geometry*.

Week 6

Lecture 14. [**M**, §18]
Continuous maps between general topological spaces. Any function $f : X \to Y$ can be made continuous by equipping X with the discrete topology or Y with the trivial topology.
Characterization at the level of a (sub)basis for the topology on the target space.
Characterization in terms of the direct image of the closure, inverse image of closed sets, in terms of neighborhoods.

Lecture 15. *(D. Freund)* [**M**, §18]
Construction of continuous functions, the Pasting Lemma.

Lecture 16. Homeomorphisms: definition, examples, topological properties.
Categories, definitions and examples: **Set**, **Mod**($\mathbb{R}$), **Top**, ordered sets: $\mathbb{N}$ and $\mathbf{Op}_X$.
Isomorphisms in a category. The isomorphisms in **Top** are the homeomorphisms.

Lecture 17. [**M**, §19]
Cartesian products of arbitrary indexed families of sets.
The box topology and the product topology: definition, comparison, bases.
A product of Hausdorff spaces is Hausdorff. The closure of a product is the product of the closures. Continuous maps into products.

Week 7

Lecture 18. [**M**, §20]
Metrizable topologies. Equivalent metrics generate the same topology. Example: the Euclidean and L^∞ metrics are equivalent and generate the product topology on $\mathbb{R}^n$.
Topologically equivalent metrics need not be equivalent: any metric is topologically equivalent to its associated standard bounded metric.
Generalization of the L^∞ topology: the uniform metric and topology on $\mathbb{R}^J$.

Lecture 19. [**M**, §20]
The uniform topology on $\mathbb{R}^J$ is intermediate between the product and box topologies.
It is metrizable if J is countable.

Lecture 20. [**M**, §21]
Sequential characterization of closure points and continuous maps in metric spaces.
Pointwise and uniform convergence a uniform limit of continuous functions is continuous.
Uniform convergence is equivalent to convergence in the uniform topology.
Functors between categories. Definition. Examples: $\mathbb{F}\mathrm{or} : \mathbf{Top} \longrightarrow \mathbf{Set}$, the 'matrix functor' $\mathbb{N} \longrightarrow \mathbf{Mod}^f(\mathbb{R})$.

Lecture 21. *(D. Freund)* [**M**, §23-24]
Definition: separations and connected spaces. Examples of connected spaces: $\mathbb{R}_\ell$, $\mathbb{Q}$. Examples of disconnected spaces: $(\mathbb{Z}, \mathcal{T}_{\text{f.c.}})$, any space with the trivial topology.
Linear continua in the order topology and their intervals are connected.
A space is connected if and only if it does not have non-trivial open and closed subsets. A connected subspace lies entirely in one component of any separation of the ambient space.
The union of connected subspaces with a common point is connected.
If A is connected and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Week 8

Lecture 22. [**M**, §23-25]
The continuous image of a connected space is connected. Finite products of connected spaces are connected, $\mathbb{R}^\infty$ is not connected in the box topology.
The Intermediate Value Theorem.
Path connectedness is a strictly stronger property than connectedness.
Connected components, every space is the disjoint union of its connected components.

Lecture 23. [**M**, §26]
Covers, Borel-Lebesgue definition of compact spaces. Examples: $\mathbb{R}$ and $(0, 1]$ are not compact, $\{0\} \cup 1/\mathbb{Z}_+$ is.
Closed subsets of compacts are compact. Compact subspaces of Hausdorff sets are closed.

Midterm 2. Characterization of T_1 spaces. Interior and boundary of $\{(x, y) \in \mathbb{R}^2\ ,\ 0 \leq y < x^2 + 1\}$.
The metric topology is the coarsest topology making the distance continuous.
The box topology on $\mathbb{R}^\omega$ is not metrizable.

Convergence in the uniform topology is equivalent to uniform convergence.
Closure of bounded and finitely supported sequences in the uniform topology.

Post-exam fun: *Furstenberg's proof of the infinitude of primes*, by N. Ezroura.
Topological groups: translation are homeomorphisms, open subgroups are closed.

Lecture 24. [**M**, §26]
Compact subsets of Hausdorff spaces are compact. Points can be separated from compacts by disjoint open sets in Hausdorff spaces. Continuous images of compact sets are compact. The Tube Lemma. Finite products of compacts are compacts.

Week 9

Lecture 25. [**M**, §27]
Segments in a totally ordered set with the least upper bound property (such as $\mathbb{R}$) are compact in the order topology. Compact sets of $\mathbb{R}^n$ in any topology associated with a metric equivalent to the ℓ^∞ metric are exactly the closed and bounded subsets.
The Extreme Value Theorem.

Lecture 26. [**M**, §28]
Féchet (limit point) compactness. Compact spaces are Fréchet compact.
Sequential compactness (Bolzano-Weierstraß property). The three notions are equivalent in metric spaces.

Lecture 27. [**M**, §29]
Locally compact Hausdorff spaces. A punctured compact Hausdorff space is locally compact and Hausdorff. Alexandrov compactification of locally compact Hausdorff spaces.

Lecture 28. [**M**, §30-34]
Generalization of sequential characterizations: nets in topological spaces.
Regular and normal spaces. First and second countability: definitions, examples and characterization in terms of closure of neighborhoods. Sequential characterizations of closure points and continuity in first countable spaces. Regular second countable spaces, metrizable spaces and compact Hausdorff spaces are normal.
The Urysohn Lemma and the Urysohn Metrization Theorem.

Week 10

Lecture 29. Topology of matrix spaces I.
Normed linear spaces over $k = \mathbb{R}$ or $\mathbb{C}$, metrics associated with norms. Equivalent norms induce equivalent metrics. In finite dimension, all norms are equivalent.
Exercise: characterizations of continuity for linear maps.
Case of $M_n(k) \subset k^{n^2}$: operator norms are submultiplicative. Continuity of operations: linear combinations, products, determinants, transposition, comatrices. The group $\mathrm{GL}(n, k)$ of invertible elements in $M_n(k)$ is a topological group. It is open in $M_n(k)$.

Lecture 30. Topology of matrix spaces II.
Density of $\mathrm{GL}(n,k)$ in $M_n(k)$ and applications: AB and BA have the same characteristic polynomial hence the same eigenvalues for any A, B in $M_n(k)$; there exists a basis of $M_n(k)$ that consists only of invertible matrices.
Connectedness properties: $\mathrm{GL}(n,\mathbb{R})$ is not connected but $\mathrm{GL}(n,\mathbb{C})$ is path connected.

- End of the course -

Problem set 1: review on sets and maps

Solution p.26

(1) **Inverse maps.**

a. Show that a map with a left (resp. right) inverse is injective (resp. surjective).

b. Give an example of a function that has a left inverse but no right inverse.

c. Give an example of a function that has a right inverse but no left inverse.

d. Can a function have more than one right inverse? More than one left inverse?

e. Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = f^{-1} = h$.

(2) Let X be a non-empty set and $m, n \in \mathbb{Z}_+$.

a. If $m \leq n$, find an injective map $f : X^m \longrightarrow X^n$.

b. Find a bijective map $g : X^m \times X^n \longrightarrow X^{m+n}$.

c. Find an injective map $h : X^n \longrightarrow X^\omega$.

d. Find a bijective map $i : X^n \times X^\omega \longrightarrow X^\omega$.

e. Find a bijective map $j : X^\omega \times X^\omega \longrightarrow X^\omega$.

f. If $A \subset B$, find an injective map $k : (A^\omega)^n \longrightarrow B^\omega$.

(3) If $A \times B$, does it follow that A and B are finite?

(4) If A and B are finite, show that the set $\mathcal{F}$ of all functions from A to B is finite.

Problem set 2: metric spaces

Solution p.28

(1) **Balls.** *No proof is required for this problem.*

a. Consider $\mathbb{Z} \times \mathbb{Z}$ equipped with the Euclidean metric.
Describe $\mathcal{B}\left((3,2), \sqrt{2}\right)$ and $\mathcal{B}_c\left((3,2), \sqrt{2}\right)$.

b. Let X be a set equipped with the discrete metric and x a point in X.
Describe the balls $\mathcal{B}(x,r)$ for all $r > 0$.

(2) **Continuous maps.**

a. Prove that the map f defined on $\mathbb{R}$ by $f(x) = x^2 + 1$ is continuous.

b. Let (E_1, d_1), (E_2, d_2), (E_3, d_3) be metric spaces and $u : E_2 \to E_3$, $v : E_1 \to E_2$ be continuous maps. Prove that $u \circ v$ is continuous.

(3) Let (E, d) be a metric space. Prove that a subset $\Omega \subset E$ is open if and only if for every point $x \in \Omega$, there exists an open ball containing x and included in Ω.

(4) Let (E, d) be a metric space and $A \subset E$ a subset. A point a in A is called *interior* if there exists $r > 0$ such that any point x in E such that $d(a, x) < r$ is in A.
The set of interior points of A is called the *interior of* A and denoted by $\overset{\circ}{A}$.

a. Prove that $\overset{\circ}{A}$ is the union of all the open balls contained in A.

b. Prove that $\overset{\circ}{A}$ is the largest open subset contained in A.

c. Can $\overset{\circ}{A}$ be empty if A is not?

Problem set 3: topological spaces

Solution p.30

(1) Let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be a family of topologies on a non-empty set X.

a. Prove that $\mathcal{I} = \bigcap_{\alpha\in A} \mathcal{T}_\alpha$ is a topology on X.

b. Prove that $\mathcal{I}$ is the finest topology that is coarser than each $\mathcal{T}_\alpha$.

(2) Let p be a prime number. Consider for $n \in \mathbb{Z}$ and a a positive integer,

$$B_a(n) = \{n + \lambda p^a \ , \ \lambda \in \mathbb{Z}\}.$$

a. Show that the family $\mathcal{B} = \{B_a(n) \ , \ n \in \mathbb{Z} \ , \ a \in \mathbb{Z}_+\}$ is a basis for a topology.

b. Is the topology generated by $\mathcal{B}$ discrete?

(3) Compare the following topologies on $\mathbb{R}$:

- $\mathcal{T}_1$: the standard topology;
- $\mathcal{T}_2$: the K-topology, with basis elements of the form

$$(a,b) \quad \text{and} \quad (a,b) \setminus \left\{ \frac{1}{n} \, , \, n \in \mathbb{Z}_+ \right\};$$

- $\mathcal{T}_3$: the finite complement topology;
- $\mathcal{T}_4$: the upper limit topology, with basis elements of the form $(a,b]$;
- $\mathcal{T}_5$: the topology with basis elements of the form $(-\infty, a)$.

(4) Let L be a straight line in $\mathbb{R}^2$. Describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Midterm 1

Solution p.32

This exam consists of 4 independent problems. Treat them in the order of your choosing, starting each problem on a new page.

Every claim you make must be **fully justified or quoted** as a result from the textbook.

Problem 1

Let $\mathcal{T}$ be the family of subsets $\mathcal{U}$ of $\mathbb{Z}_+$ satisfying the following property:

If n is in $\mathcal{U}$, then any divisor of n belongs to $\mathcal{U}$.

1. Give two different examples of elements of $\mathcal{T}$ containing 24 (not including $\mathbb{Z}_+$).

2. Verify that $\mathcal{T}$ is a topology on $\mathbb{Z}_+$.

3. Is $\mathcal{T}$ the discrete topology?

Problem 2

Let (E, d) be a metric space.

1. Recall the definition of the metric topology and prove that open balls form a basis.

2. Assume that ρ is a second metric on E such that, for every $x, y \in E$,

$$\frac{1}{2}d(x, y) \leq \rho(x, y) \leq 2d(x, y).$$

Compare the topologies generated by d and ρ.

Problem 3

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X.

1. Verify that $\mathcal{T}_1 \cup \mathcal{T}_2$ is a subbasis for a topology.

From now on, $\mathcal{T}_1 \vee \mathcal{T}_2$ denotes the topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$.

2. Describe $\mathcal{T}_1 \vee \mathcal{T}_2$ when $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

3. Compare $\mathcal{T}_1 \vee \mathcal{T}_2$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ in general.

4. Let $\mathcal{T}$ be a topology on X that is finer than $\mathcal{T}_1$ and $\mathcal{T}_2$.
Prove that $\mathcal{T}$ is finer than $\mathcal{T}_1 \vee \mathcal{T}_2$.

Problem 4

1. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \left\{ x \ , \ \frac{1}{2} < |x| < 1 \right\}$$
$$B = \left\{ x \ , \ \frac{1}{2} < |x| \leq 1 \right\}$$
$$C = \left\{ x \ , \ \frac{1}{2} \leq |x| < 1 \right\}$$
$$D = \left\{ x \ , \ 0 < |x| < 1 \text{ and } \frac{1}{x} \in \mathbb{Z}_+ \right\}$$

2. Let $X = \mathbb{R}_\ell \times \mathbb{R}_u$ where $\mathbb{R}_\ell$ denotes the topology with basis consisting of all intervals of the form $[a, b)$ and $\mathbb{R}_u$ denotes the topology with basis consisting of all intervals of the form $(c, d]$.

Describe the topology induced on the plane curve Γ with equation $y = e^x$.

Problem set 4: closed sets and limit points

Solution p.35

(1) Prove the following result:
Theorem Let X be a set and $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ a map such that, for any $A, B \subset X$,
(i) $\gamma(\varnothing) = \varnothing$;
(ii) $A \subset \gamma(A)$;
(iii) $\gamma(\gamma(A)) = \gamma(A)$;
(iv) $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$.
Then the family $\{X \setminus \gamma(A)\,,\ A \in \mathcal{P}(X)\}$ is a topology in which $\overline{A} = \gamma(A)$.
Hint: it might be useful to prove that $A \subset B \Rightarrow \gamma(A) \subset \gamma(B)$.

(2) *The questions in this problem are independent.*
(a) Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x)\ ,\ x \in X\}$ is closed in $X \times X$.
(b) Determine the accumulation points of the subset $\left\{\frac{1}{m} + \frac{1}{n}\ ,\ m, n \in \mathbb{Z}_+\right\}$ of $\mathbb{R}$.

(3) The boundary of a subset A in a topological space X is defined by
$$\partial A = \bar{A} \cap \overline{X \setminus A}.$$
(a) Show that $\bar{A} = \mathring{A} \sqcup \partial A$[2].
(b) Show that $\partial A = \varnothing$ if and only if A is open and closed.
(c) Show that U is open if and only if $\partial U = \bar{U} \setminus U$.
(d) If U is open, is it true that $U = \mathring{\bar{U}}$?

(4) Find the boundary and interior of each of the following subsets of $\mathbb{R}^2$.
(a) $A = \{(x, y)\ ,\ y = 0\}$
(b) $B = \{(x, y)\ ,\ x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \mathbb{Q} \times \mathbb{R}$
(e) $E = \{(x, y)\ ,\ 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{(x, y)\ ,\ x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

[2] The *disjoint union* symbol $\sqcup$ is used to indicate that the sets in the union have empty intersection.

Problem set 5: continuous maps, the product topology

Solution p.40

(1) (a) Consider $\mathbb{Z}_+$ equipped with the topology in which open sets are the subsets U such that if n is in U, then any divisor of n belongs to U. Give a necessary and sufficient condition for a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ to be continuous.

(b) Let $\chi_{\mathbb{Q}}$ be the indicator of $\mathbb{Q}$. Prove that the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(x) = x \cdot \chi_{\mathbb{Q}}(x)$ is continuous at exactly one point.

(2) Let X and Y be topological spaces. If A is a subset of either, we denote by A' the sets of accumulation points of A and by ∂A its boundary.
Let $f : X \longrightarrow Y$ be a map. Determine the implications between the following statements:

(i) f is continuous.

(ii) $f(A') \subset (f(A))'$ for any $A \subset X$.

(iii) $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for any $B \subset Y$.

(3) Let X and Y be topological spaces, and assume Y Hausdorff. Let A be a subset of X and f_1, f_2 continuous maps from the closure $\bar{A}$ to Y.
Prove that if f_1 and f_2 restrict to the same function $f : A \to Y$, then $f_1 = f_2$.

(4) Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $X = \prod_{\alpha \in J} X_\alpha$.

(a) Give a necessary and sufficient condition for a sequence $\{u_n\}_{n \in \mathbb{Z}_+}$ to converge in X equipped with the product topology.

(b) Does the result hold if X is equipped with the box topology?

Problem set 6: metrizable spaces

Solution p.43

(1) Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. For $x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega$ and $0 < \varepsilon < 1$, let

$$P(x,\varepsilon) = \prod_{n\in\mathbb{Z}_+} (x_n - \varepsilon, x_n + \varepsilon).$$

(a) Compare $P(x,\varepsilon)$ with $B_{\bar{\rho}}(x,\varepsilon)$.
(b) Is $P(x,\varepsilon)$ open in the uniform topology?
(c) Show that $B_{\bar{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} P(x,\delta)$.

(2) We denote by $\ell^2(\mathbb{Z}_+)$ the set of square-summable real-valued sequences, that is,

$$\ell^2(\mathbb{Z}_+) = \left\{ x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega \quad , \quad \sum_{n\geq 1} x_n^2 \quad \text{converges} \right\}.$$

We admit that the formula

$$d(x,y) = \left(\sum_{n\geq 1} (x_n - y_n)^2 \right)^{1/2}$$

defines a metric on $\ell^2(\mathbb{Z}_+)$.

(a) Compare the metric topology induced by d on $\ell^2(\mathbb{Z}_+)$ with the restrictions of the box and uniform topologies from $\mathbb{R}^\omega$.
(b) Let $\mathbb{R}^\infty$ denote the subset of $\ell^2(\mathbb{Z}_+)$ consisting of sequences that have finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\ell^2(\mathbb{Z}_+)$.

(3) Let X be a topological space, Y a metric space and assume that $(f_n)_{n\geq 0}$ is a sequence of continuous functions that converges uniformly to $f : X \longrightarrow Y$.
Let $(x_n)_{n\geq 0}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

(4) **Ultrametric spaces** (*non-mandatory*)**.**

Let X be a set equipped with a map $d : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$,

(1) $d(x, y) \geq 0$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) = 0 \Leftrightarrow x = y$
(4) $d(x, z) \leq \max(d(x, y), d(y, z))$

(a) Prove that d is a distance.
(b) Let B be an open ball for d. Prove that $B = B(y, r)$ for every element $y \in B$ for some $r > 0$.
(c) Prove that closed balls are open and open balls are closed in the topology induced by d.

Midterm 2: in-class examination

Solution p.47

This exam consists of 5 independent problems. You may treat them in the order of your choosing, starting each problem on a new page.

Every claim you make must be **fully justified or quoted** as a result from the textbook.

Problem 1

1. Show that a topological space is T_1 if and only if for any pair of distinct points, each has a neighborhood that does not contain the other[3].

2. Determine the interior and the boundary of the set

$$\Xi = \left\{(x, y) \in \mathbb{R}^2 \ , \ 0 \leq y < x^2 + 1\right\}$$

where $\mathbb{R}^2$ is equipped with its ordinary Euclidean topology.

Problem 2

Let E be a set with a metric d and $\mathcal{T}_d$ the corresponding metric topology on E.

1. Prove that the map $d : (E, \mathcal{T}_d) \times (E, \mathcal{T}_d) \longrightarrow \mathbb{R}$ is continuous.

2. Let $\mathcal{T}$ be a topology on E, such that $d : (E, \mathcal{T}) \times (E, \mathcal{T}) \longrightarrow \mathbb{R}$ is continuous. Prove that $\mathcal{T}$ is finer than $\mathcal{T}_d$.

Hint: it might be helpful to consider sets of the form $d^{-1}\left((-\infty, r)\right)$ for $r > 0$.

[3] A topological space is said T_1 if all its singletons are closed.

Problem 3

The purpose of this problem is to prove that the box topology on $\mathbb{R}^\omega$ is not metrizable.

1. Recall the definition of the box topology on $\mathbb{R}^\omega$.

Denote by $\mathbf{0}$ the sequence constantly equal to 0 and let

$$P = (0, +\infty)^\omega = \prod_{n \geq 1} (0, +\infty)$$

be the subset of positive sequences.

2. Verify that $\mathbf{0}$ belongs to $\bar{P}$.

3. Prove that no sequence $(p_n)_{n \geq 1} \in P^\omega$ converges to $\mathbf{0}$ in the box topology.

4. Conclude.

Problem 4

1. Let X be a set.

(a) Recall the definition of the uniform topology on $\mathbb{R}^X$.

(b) Recall the definition of uniform convergence for a sequence of functions f_n in $\mathbb{R}^X$.

2. Prove that a sequence in $\mathbb{R}^X$ converges uniformly if and only if it converges for the uniform topology.

Problem 5

Consider the space $\mathbb{R}^\omega$ of real-valued sequences, equipped with the uniform topology.

1. Prove that the subset B of bounded sequences is closed in $\mathbb{R}^\omega$ for the uniform topology.

2. Let $\mathbb{R}^\infty$ denote the subset of sequences with finitely many non-zero terms.
Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ for the uniform topology.

Midterm 2: take-home examination

Solution p.51

This is an individual assignment. You may use the text and class notes but no other source or outside help.

Problem 1

1. Determine the connected components of $\mathbb{R}^\omega$ in the product topology.

2. Consider $\mathbb{R}^\omega$ equipped with the uniform topology.

(a) Prove that x is in the same connected component as $\mathbf{0}$ if and only if x is bounded.

(b) Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the uniform topology.

3. Consider $\mathbb{R}^\omega$ equipped with the box topology.

(a) Let $x, y \in \mathbb{R}^\omega$ be such that $x - y \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$.
Prove that there exists a homeomorphism

$$\varphi : \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$$

such that $(\varphi(x)_n)_{n\in\mathbb{Z}_+}$ is a bounded sequence and $(\varphi(y)_n)_{n\in\mathbb{Z}_+}$ is unbounded.

Hint: given $u \in \mathbb{R}^\omega$, it might be helpful to consider the sequence v defined by

$$v_n = \begin{cases} u_n - x_n & \text{if } x_n = y_n \\ \frac{u_n - x_n}{y_n - x_n} & \text{if } x_n \neq y_n \end{cases} .$$

(b) Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the box topology.

Problem 2

Let F be a functor between categories $\mathcal{C}$ and $\mathcal{C}'$. A functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ is said to be a *left adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(X), Y) \cong \mathrm{Hom}_{\mathcal{C}'}(X, F(Y))$$

for all objects $X \in \underline{\mathcal{C}'}$ and $Y \in \underline{\mathcal{C}}$. Similarly, G is called a *right adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{C}'}(F(X), Y)$$

for all objects $X \in \underline{\mathcal{C}}$ and $Y \in \underline{\mathcal{C}'}$.

Recall that the forgetful functor $\mathbb{F} : \mathbf{Top} \longrightarrow \mathbf{Set}$ is defined by

- $\mathbb{F}((X, \mathcal{T})) = X$ for any set X equipped with a topology $\mathcal{T}$;
- $\mathbb{F}(f) = f$ for any continuous map $f : X \longrightarrow Y$.

If X is a set, let $\mathbb{G}(X)$ denote the topological space obtained by endowing X with the trivial topology $\mathcal{T}_{\text{triv.}} = \{X, \emptyset\}$:

$$\mathbb{G}(X) = (X, \mathcal{T}_{\text{triv.}}).$$

If f is a map between sets, define in addition $\mathbb{G}(f) = f$.

1. Verify that $\mathbb{G}$ is a functor.

2. Prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$.

3. Find a left adjoint for $\mathbb{F}$.

Problem set 7: connectedness and compactness

Solution p.55

(1) Let U be an open connected subspace of $\mathbb{R}^2$ and $a \in U$.

(a) Prove that the set of points $x \in U$ such that there is a path $\gamma : [0,1] \longrightarrow U$ with $\gamma(0) = a$ and $\gamma(1) = x$ is open and closed in U.

(b) What can you conclude?

(2) Let X be a topological space and $Y \subset X$ a connected subspace.

(a) Are $\mathring{Y}$ and ∂Y necessarily connected?

(b) Does the converse hold?

(3) Let (E, d) be a metric space.

(a) Prove that every compact subspace of E is closed and bounded.

(b) Give an example of metric space in which closed bounded sets are not necessarily compact.

(4) (a) Prove that the Alexandrov compactification of $\mathbb{R}$ is homeomorphic to the unit circle

$$\mathrm{S}^1 = \{(x, y) \in \mathbb{R}^2 \ , \ x^2 + y^2 = 1\}.$$

(b) Verify that $\mathbb{Z}_+ \subset \mathbb{R}$ is a locally compact Hausdorff space.

(c) Prove that the Alexandrov compactification of $\mathbb{Z}_+$ is homeomorphic to

$$\left\{ \frac{1}{n} \ , \ n \in \mathbb{Z}_+ \right\} \cup \{0\}.$$

Final examination

Solution p.58

This exam consists of 6 independent problems. Your may treat them in the order of your choosing, starting each problem on a new page.

Problem 1

1. Let X be a Hausdorff space and K_1, K_2 disjoint compact subsets of X.
Prove that there exist disjoint open sets U_1 and U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.

2. Let X be a discrete topological space. Describe the compact subsets of X.

Problem 2

A topological space is said *totally disconnected* if its only connected subspaces are singletons.

1. Prove that a discrete space is totally disconnected.

2. Does the converse hold?

Problem 3

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces; let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$.

1. In $\prod_{\alpha \in J} X_\alpha$ equipped with the product topology, prove that

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

2. Does the result hold if $\prod_{\alpha \in J} X_\alpha$ carries the box topology?

Problem 4

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

Problem 5

Let (E, d) be a metric space. An *isometry of* E is a map $f : E \longrightarrow E$ such that

$$d\left(f(x), f(y)\right) = d(x, y)$$

for all $x, y \in E$.

1. Prove that any isometry is continuous and injective.

Assume from now on that E is compact and f an isometry. We want to prove that f is surjective. Assume to the contrary the existence of $a \notin f(E)$.

2. Prove that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subset E \setminus f(E)$.

3. Consider the sequence defined by $x_1 = a$ and $x_{n+1} = f(x_n)$. Prove that

$$d(x_n, x_m) \geq \varepsilon$$

for $n \neq m$ and derive a contradiction.

4. Prove that an isometry of a compact metric space is a homeomorphism.

Problem 6

Let X be a set, $\mathscr{P}(X)$ the set of subsets of X and $\iota : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ a map satisfying:

$$\begin{array}{ll} (1) & \iota(X) = X \\ (2) & \iota(A) \subset A \\ (3) & \iota \circ \iota(A) = \iota(A) \\ (4) & \iota(A \cap B) = \iota(A) \cap \iota(B) \end{array}$$

for all $A, B \subset X$.

1. Check that $A \subset B \;\Rightarrow\; \iota(A) \subset \iota(B)$ for $A, B \subset X$.

2. Prove that the family $\mathcal{T} = \{\iota(A)\ ,\ A \in \mathscr{P}(X)\}$ is a topology on X.

3. Prove that, in this topology, $\mathring{A} = \iota(A)$ for all $A \subset X$.

Problem set 1: review on sets and maps - Elements of solution

This is supposedly well-known material. We simply indicate solutions and hints.

(1) **Inverse maps.**

a. Show that a map with a left (resp. right) inverse is injective (resp. surjective).

If $f \circ g = Id$, then $g(x_1) = g(x_2) \;\Rightarrow\; x_1 = f(g(x_1)) = f(g(x_2)) = x_2$ so g is injective and $y = f(g(y))$ for all y so f is surjective.

b. Give an example of function with a left inverse but no right inverse.
c. Give an example of function with a right inverse but no left inverse.

Consider a two-point set $\{a, b\}$ and a singleton $\{x\}$. Then $\varphi_a : x \mapsto a$ and $\varphi_b : x \mapsto b$ are both right inverses to the only map ψ from $\{a, b\}$ to $\{x\}$, which is not injective hence cannot have a left inverse.
Since, $\psi \circ \varphi_a = \psi \circ \varphi_b = \mathrm{Id}_{\{x\}}$, both φ_a and φ_b have a left inverse, but they have no right inverse as the only candidate would be ψ and $\varphi_a \circ \psi \neq \mathrm{Id}_{\{a,b\}}$.

d. Can a function have more than one right inverse? More than one left inverse?

The function ψ defined above has multiple right inverses.
Consider the map $\lambda : \{a, b\} \longrightarrow \{u, v, w\}$ defined by $\lambda(a) = u$ and $\lambda(b) = v$.
The maps $\mu_a, \mu_b : \{u, v, w\} \longrightarrow \{a, b\}$ respectively defined by

$$\mu_a(u) = \mu_b(u) = a \quad , \quad \mu_a(v) = \mu_b(v) = b$$

and

$$\mu_a(w) = a \quad , \quad \mu_b(w) = b$$

are both left inverses of λ.

e. Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = f^{-1} = h$.

It follows from **a.** that f is bijective. Therefore, $g \circ f = \mathrm{Id}$ implies $g = f^{-1}$ while $f \circ h = \mathrm{Id}$ implies $h = f^{-1}$.

(2) **Let X be a non-empty set and $m, n \in \mathbb{Z}_+$.**

a. If $m \leq n$, find an injective map $f : X^m \longrightarrow X^n$.

Consider $f(x_{,1}, \ldots, x_m) = (x_1, \ldots, x_m, \overbrace{x_0, \ldots, x_0}^{n-m})$ for some $x_0 \in X$.

b. Find a bijective map $g : X^m \times X^n \longrightarrow X^{m+n}$.

Consider $g\left((x_1, \ldots, x_m), (y_1, \ldots, y_n)\right) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$

c. Find an injective map $h : X^m \longrightarrow X^\omega$.

d. Find a bijective map $i : X^m \times X^\omega \longrightarrow X^\omega$.

Consider 'infinite versions' of the functions f and g defined above.

e. Find a bijective map $j : X^\omega \times X^\omega \longrightarrow X^\omega$.

Consider $j : \left((x_\ell)_{\ell \in \mathbb{Z}_+}, (y_\ell)_{\ell \in \mathbb{Z}_+}\right) \longmapsto (z_\ell)_{\ell \in \mathbb{Z}_+}$ with $z_{2\ell-1} = x_\ell$ and $z_{2\ell} = y_\ell$.

f. If $A \subset B$, find an injective map $k : (A^\omega)^n \longrightarrow B^\omega$.

Consider $k : \left({}^1x_\ell)_{\ell \in \mathbb{Z}_+}, \ldots, ({}^nx_\ell)_{\ell \in \mathbb{Z}_+}\right) \mapsto ({}^1x_1, \ldots, {}^nx_1, , \ldots, {}^1x_\ell, \ldots, {}^nx_\ell, \ldots)$.

(3) **If $A \times B$, does it follow that A and B are finite?**

No: $\varnothing \times \mathbb{Z}_+ = \varnothing$. However, if both A and B are non-empty, and one of them is infinite, say B, then $A \times B$ contains some $\{a\} \times B$ which is in bijection with B, hence infinite.

(4) **If A and B are finite, show that the set $\mathcal{F} = \{f : A \longrightarrow B\}$ is finite.**
Let $A = \{a_1, \ldots, a_n\}$. Then $f \longmapsto (f(a_1), \ldots, f(a_n))$ is a bijection from $\mathcal{F}$ to B^n, which has cardinal $|B|^n$, hence is finite.

Problem set 2: metric spaces - Elements of solution

(1) **a. Consider $\mathbb{Z} \times \mathbb{Z}$ equipped with the Euclidean metric. Describe $\mathcal{B}\left((3,2), \sqrt{2}\right)$ and $\mathcal{B}_c\left((3,2), \sqrt{2}\right)$.**

One can enumerate the elements:

$$\mathcal{B}\left((3,2), \sqrt{2}\right) = \{(2,2); (3,1); (3,2); (3,3); (4,2)\}.$$

and

$$\mathcal{B}_c\left((3,2), \sqrt{2}\right) = \mathcal{B}\left((3,2), \sqrt{2}\right) \cup \{(2,1); (2,3); (4,1); (4,3)\}$$

b. Let X be a set equipped with the discrete metric and x in X. Describe the balls $\mathcal{B}(x,r)$ for all $r > 0$.

By definition, $\mathcal{B}(x,r) = \{x\}$ for $0 < r \leq 1$ and $\mathcal{B}(x,r) = X$ for $r > 1$.

(2) **Continuous maps.**

a. Prove that the map f defined on $\mathbb{R}$ by $f(x) = x^2 + 1$ is continuous.

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Note that if $a - 1 \leq x \leq a + 1$, then $|x + a| \leq 2|a| + 1$. Therefore, since $|f(x) - f(a)| = |x - a||x + a|$, we get, for $x \in [a - 1, a + 1]$,

$$|f(x) - f(a)| \leq |x - a|(2|a| + 1)$$

and it suffices to choose $|x - a| < \min\{\frac{\varepsilon}{2|a|+1}, 1\}$ to guarantee $|f(x) - f(a)| < \varepsilon$.

b. Let E_1, E_2, E_3 be metric spaces and $u : E_2 \to E_3$, $v : E_1 \to E_2$ be continuous maps. Prove that $u \circ v$ is continuous.

Let Ω be open in E_3 and apply Theorem [MC] twice: $u^{-1}(\Omega)$ is open in E_2 by continuity of u and $(u \circ v)^{-1}(\Omega) = v^{-1}(u^{-1}(\Omega))$ is open by continuity of v.

(3) **Let (E, d) be a metric space. Prove that a subset $\Omega \subset E$ is open if and only if for every point $x \in \Omega$, there exists an open ball containing x and included in Ω.**

The definition seen in class for open sets in a metric space differs only by the fact that it requires the ball to be centered at the point considered. Therefore, open sets trivially satisfy the property.
Observe that if a point x is included in a ball $\mathcal{B}(a,r)$, the triangle inequality implies that $\mathcal{B}(x, r - d(a,x))$ is included in $\mathcal{B}(a,r)$. The converse follows.

(4) **Let (E, d) be a metric space and $A \subset E$. A point a in A is called *interior* if there exists $r > 0$ such that any point x in E such that $d(a, x) < r$ is in A. The set $\overset{o}{A}$ of interior points of A is called the *interior of* A.**

a. Prove that $\overset{o}{A}$ is the union of all the open balls contained in A.

Let $\dot{A}$ be the union of all the open balls contained in A and let a be in $\dot{A}$. By definition of $\dot{A}$ and the argument used in (3), there exists a ball $\mathcal{B}(a, r)$ included in A, so $\dot{A} \subset \overset{o}{A}$. Conversely, let a be in $\overset{o}{A}$. By definition, there exists $r > 0$ such that $\mathcal{B}(a, r) \subset A$ so $a \in \dot{A}$, hence the result.

b. Prove that $\overset{o}{A}$ is the largest open subset contained in A.

First, $\overset{o}{A}$ is open as the union of open subsets, as proved in **a.** We argue by contradiction: assume the existence of Ω open such that $\overset{o}{A} \subsetneq \Omega \subset A$. Let $x \in \Omega \setminus \overset{o}{A}$. Since $x \notin \overset{o}{A}$, no ball $\mathcal{B}(x, r)$ with $r > 0$ is contained in A. Since Ω is open, it must contain such a ball, which contradicts the assumption that $\Omega \subset A$.

c. Can $\overset{o}{A}$ be empty if A is not?

Yes: consider for instance $A = \mathbb{Z}$ in $E = \mathbb{R}$, or any strict linear subspace of $\mathbb{R}^n$ and observe that every ball centered at the origin must contain a basis.

Problem set 3: topological spaces - Elements of solution

(1) **Let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be a family of topologies on a non-empty set X.**

a. Prove that $\mathcal{I} = \bigcap_{\alpha\in A} \mathcal{T}_\alpha$ is a topology on X.

Direct verification, using the definition and the fact that $\mathcal{I} \subset \mathcal{T}_\alpha$ for each α.

b. Prove that $\mathcal{I}$ is the finest topology that is coarser than each $\mathcal{T}_\alpha$.

By construction, $\mathcal{I}$ is coarser than (included in) each $\mathcal{T}_\alpha$. Let $\mathcal{J}$ be a finer topology than $\mathcal{I}$ and consider $U \in \mathcal{J} \setminus \mathcal{I}$. Since $U \notin \mathcal{I}$, there exists some $\alpha_0 \in A$ such that $U \notin \mathcal{T}_{\alpha_0}$, so that $\mathcal{J}$ cannot be coarser than $\mathcal{T}_{\alpha_0}$.

(2) **Let p be a prime number. Consider for $n \in \mathbb{Z}$ and a a positive integer,**

$$B_a(n) = \{n + \lambda p^a \ , \ \lambda \in \mathbb{Z}\}.$$

a. Show that $\mathcal{B} = \{B_a(n) \ , \ n \in \mathbb{Z} \ , \ a \in \mathbb{Z}_+\}$ is a basis for a topology.

Every $n \in \mathbb{Z}$ is in $B_a(n)$ so the covering property is satisfied. The other property follows from the observation that if $B_a(n) \cap B_b(m)$ is either empty of of the form $B_{\max(a,b)}(q)$.

b. Is the topology generated by $\mathcal{B}$ discrete?

No: no finite subset of $\mathbb{Z}$ is open in this topology.

(3) **Compare the following topologies on $\mathbb{R}$:**

- **$\mathcal{T}_1$: the standard topology;**
- **$\mathcal{T}_2$: the K-topology, with basis elements of the form (a, b) and $(a, b) \setminus \left\{\frac{1}{n} \ , \ n \in \mathbb{Z}_+\right\}$;**
- **$\mathcal{T}_3$: the finite complement topology;**
- **$\mathcal{T}_4$: the upper limit topology, with basis elements of the form $(a, b]$;**
- **$\mathcal{T}_5$: the topology with basis elements of the form $(-\infty, a)$.**

Comparison of $\mathcal{T}_1$ with $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$, $\mathcal{T}_5$

It is proved in Lemma 13.4 that $\mathcal{T}_1 \subsetneq \mathcal{T}_2$.
To compare $\mathcal{T}_1$ with $\mathcal{T}_3$, observe that any subset of $\mathbb{R}$ with finite complement is of the form $(-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_n, +\infty)$, that is a union of basis elements for $\mathcal{T}_1$. Therefore, $\mathcal{T}_3 \subset \mathcal{T}_1$ and the inclusion is strict since any union of finitely complemented set is finitely complemented: $\mathcal{T}_3 \subsetneq \mathcal{T}_1$.
The argument given in Lemma 13.4 for the lower limit topology can be adapted to show that $\mathcal{T}_1 \subsetneq \mathcal{T}_4$.
Finally, $\mathcal{T}_5 \subset \mathcal{T}_1$ follows from the observation that $(-\infty, a) = \bigcup_{z<a}(z, a)$. The inclusion is strict since no interval of the form (a, b) with a finite lies in $\mathcal{T}_5$.

Comparison of $\mathcal{T}_2$ with $\mathcal{T}_3$, $\mathcal{T}_4$, $\mathcal{T}_5$

By transitivity of inclusions, $\mathcal{T}_3 \subsetneq \mathcal{T}_2$ and $\mathcal{T}_5 \subsetneq \mathcal{T}_2$.
We shall now prove that $\mathcal{T}_2 \subsetneq \mathcal{T}_4$ (compare to Lemma 13.4 and Problem 13.6). The inclusion relies on the observation that $(a, b) = \bigcup_{a<\beta<b}(a, \beta]$, which implies that both types of basis elements for $\mathcal{T}_2$ are union of basis elements of $\mathcal{T}_4$. The inclusion is strict: consider for instance $(0, 2] \in \mathcal{T}_4$. No basis element of $\mathcal{T}_2$ can contain 2 and be included in $(0, 2]$.

Comparison of $\mathcal{T}_3$ with $\mathcal{T}_4$, $\mathcal{T}_5$

By transitivity again, $\mathcal{T}_3 \subsetneq \mathcal{T}_4$.
There is no inclusion relation between $\mathcal{T}_3$ and $\mathcal{T}_5$: as observed before, no union of basis elements of $\mathcal{T}_3$ can have infinite complement as $(-\infty, 0) \in \mathcal{T}_5$ does, so $\mathcal{T}_5 \not\subset \mathcal{T}_3$. Conversely, $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$ is a basis element for $\mathcal{T}_3$ that does not belong to $\mathcal{T}_5$ in which all open sets are convex.

Comparison of $\mathcal{T}_4$ with $\mathcal{T}_5$

By transitivity, $\mathcal{T}_5 \subsetneq \mathcal{T}_4$.

The relations are summed up in the following diagram, in which all the inclusions are strict:

$$\begin{array}{ccccccc} & & \mathcal{T}_5 & & & & \\ & & \cap & & & & \\ \mathcal{T}_3 & \subset & \mathcal{T}_1 & \subset & \mathcal{T}_2 & \subset & \mathcal{T}_4 \end{array}$$

(4) **Let L be a straight line in $\mathbb{R}^2$. Describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.**

Lemma 16.1 states that a basis for the topology in question is given by considering intersections of L with basis elements of the chosen topology on $\mathbb{R}^2$.

- In the case of $L \subset \mathbb{R}_\ell \times \mathbb{R}$, they are of the form $L \cap [a, b) \times (c, d)$. Identifying[4] L with $\mathbb{R}$, we see that these subsets are of the form (α, β) if L is vertical and include those of the form $[\alpha, \beta)$ otherwise. Therefore, if L is vertical, the subspace topology is the standard topology on the real line, while if L is slanted, it carries the lower limit topology.
- In the case of $L \subset \mathbb{R}_\ell \times \mathbb{R}$, the same line of reasoning shows that L receives the lower limit topology if it is vertical or has a non-negative slope. If the slope is negative, the sets $L \cap [a, b) \times (c, d)$ include those of the form $[\alpha, \beta]$ and conversely, any segment can be obtained as such an intersection. Since singletons are special cases ($\{\alpha\} = [\alpha, \alpha]$), it follows that the induced topology on L is discrete.

[4] A rigorous way to do this will be explained later in the course.

Midterm 1 - Elements of solution

[M] refers to *Topology*, 2nd ed. by J. Munkres.

Problem 1

Let $\mathcal{T}$ be the family of subsets $\mathcal{U}$ of $\mathbb{Z}_+$ satisfying the following property:
If n is in $\mathcal{U}$, then any divisor of n belongs to $\mathcal{U}$.

1. Give two different examples of elements of $\mathcal{T}$ containing 24.

The prime factorization of 24 is $24 = 2^3 \cdot 3$ so divisors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24. Any element of $\mathcal{T}$ containing 24 must contain all of these. If we adjoin another number to this list, we must also include all of its divisors. For instance, any open set containing 10 will also contain 5. Examples are $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 24\}$ or $\{1, 2, 3, 4, 6, 8, 12, 17, 24\}$.

2. Verify that $\mathcal{T}$ is a topology on $\mathbb{Z}_+$.

We verify the three axioms. (O1) The empty set trivially belongs to $\mathcal{T}$ and $\mathbb{Z}_+$ contains the divisors of any integer so it is open too.
(O2) Let $\{U_\alpha\}_{\alpha \in A}$ be a family of elements of $\mathcal{T}$ and $U = \bigcup_{\alpha \in A} U_\alpha$. Let $n \in U$. There exists $\alpha_0 \in A$ such that $n \in U_{\alpha_0}$ so all the divisors of n belong to U_{α_0} open, hence to U. This means that U is open by definition.
(O3) Let $\{U_i\}_{1 \leq i \leq p}$ be a family of elements of $\mathcal{T}$ and $U = \bigcap_{i=1}^{p} U_i$. Let $n \in U$. For every $1 \leq i \leq p$, the integer n belongs to U_i so every divisor of n belongs to all the U_i's, hence to U which is therefore open. This proves that $\mathcal{T}$ is a topology.

3. Is $\mathcal{T}$ the discrete topology?

Since 1 is a divisor of any integer, every non-empty open set must contain 1. Therefore, there exists subsets of $\mathbb{Z}_+$ that are not open and $\mathcal{T}$ is not discrete.

Problem 2

Let (E, d) be a metric space.

1. Recall the definition of the metric topology and prove that open balls form a basis for that topology.

A subset Ω of E is open if any point of Ω is contained in an open ball included in Ω. By [M, Lemma 13.2], this implies that balls form a basis for the metric topology.

2. Assume that ρ is a second metric on E such that, for every $x, y \in E$,

$$\frac{1}{2}d(x,y) \leq \rho(x,y) \leq 2d(x,y).$$

Compare the topologies generated by d and ρ.

Denote by $\mathcal{T}_d$ and $\mathcal{T}_\rho$ the topologies associated with the metrics d and ρ. We shall prove that $\mathcal{T}_\rho$ is finer than $\mathcal{T}_d$.
Let $B_d(a,r)$ be a ball and $x \in B_d(a,r)$. Since $B_d(a,r)$ is open for $\mathcal{T}_d$, there exists a radius $r' > 0$ such that $B_d(x,r') \subset B_d(a,r)$. The inequalities satisfied by d and ρ imply that

$$x \in B_\rho(x, \frac{r'}{2}) \subset B_d(x,r').$$

Indeed, if $\rho(x,y) < \frac{r'}{2}$, then $\frac{1}{2}d(x,y) < \frac{r'}{2}$ so $y \in B_d(x,r')$.
Lemma 13.3 in [M] then implies that $\mathcal{T}_d \subset \mathcal{T}_\rho$. The converse inclusion can be proved by the same argument, using the other inequality satisfied by ρ and d and we can conclude that $\mathcal{T}_d = \mathcal{T}_\rho$.

Problem 3

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X.

1. Verify that $\mathcal{T}_1 \cup \mathcal{T}_2$ is a subbasis for a topology.

Any $x \in X$ is included in an element of $\mathcal{T}_1$ (e.g X) hence of $\mathcal{T}_1 \cup \mathcal{T}_2$, which is therefore a subbasis for a topology.

From now on, $\mathcal{T}_1 \vee \mathcal{T}_2$ denotes the topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$.

2. Describe $\mathcal{T}_1 \vee \mathcal{T}_2$ when $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}_2$ so $\mathcal{T}_1 \vee \mathcal{T}_2$ is the topology generated by $\mathcal{T}_2$, that is, $\mathcal{T}_2$ itself.

3. Compare $\mathcal{T}_1 \vee \mathcal{T}_2$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ in general.

By definition, $\mathcal{T}_1 \vee \mathcal{T}_2$ contains $\mathcal{T}_1$ and $\mathcal{T}_2$ so it is finer than both.

4. Let $\mathcal{T}$ be a finer topology than $\mathcal{T}_1$ and $\mathcal{T}_2$. Prove that $\mathcal{T}$ is finer than $\mathcal{T}_1 \vee \mathcal{T}_2$.

Any element of $\mathcal{T}_1 \vee \mathcal{T}_2$ is the union of subsets of X of the form $U = \bigcap_{i=1}^{n} U_i$ with $U_i \in \mathcal{T}_1 \cup \mathcal{T}_2$ for each i. Such a U is an intersection of elements of $\mathcal{T}_1$ and of $\mathcal{T}_2$, all of which belong to $\mathcal{T}$ assumed finer so U belongs to $\mathcal{T}$. Since $\mathcal{T}$ is stable under unions, it follows that $\mathcal{T}_1 \vee \mathcal{T}_2 \subset \mathcal{T}$. In other words, $\mathcal{T}_1 \vee \mathcal{T}_2$ is the coarsest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$

Problem 4

1. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$A = \left\{x \ , \ \frac{1}{2} < |x| < 1\right\} = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ is open in $\mathbb{R}$ as the union of two basis elements. It is also open in Y by definition.

$B = \left\{x \ , \ \frac{1}{2} < |x| \leq 1\right\} = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ is not open in $\mathbb{R}$ since no open interval containing 1 is included in B. It is open in Y as the intersection of Y with $(-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$, open in $\mathbb{R}$.

$C = \left\{x \ , \ \frac{1}{2} \leq |x| < 1\right\} = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$ is not open in $\mathbb{R}$ since no open interval containing $\frac{1}{2}$ is included in C. It is not open in Y for the same reason: no intersection of Y with an open interval containing $\frac{1}{2}$ is included in C.

$D = \left\{x \ , \ 0 < |x| < 1 \text{ and } \frac{1}{x} \in \mathbb{Z}_+\right\} = \left\{\frac{1}{n} \ , \ n \in \mathbb{Z}_+ \ , \ n \geq 2\right\}$ is neither open in $\mathbb{R}$ nor in Y. The argument used for C can be used without modification.

2. Let $X = \mathbb{R}_\ell \times \mathbb{R}_u$ where $\mathbb{R}_\ell$ denotes the topology with basis consisting of all intervals of the form $[a, b)$ and $\mathbb{R}_u$ denotes the topology with basis consisting of all intervals of the form $(c, d]$. Describe the topology induced on the plane curve Γ with equation $y = e^x$.

A basis for the subspace topology on Γ is given by the sets $[a, b) \times (c, d] \cap \Gamma$. Singletons are of this form: $\{(x, e^x)\} = [x, x+1) \times (e^x - 1, e^x] \cap \Gamma$ so the induced topology is discrete.

Problem set 4: closed sets and limit points - Elements of solution

(1) **Prove the following result:**
Theorem Let X be a set and $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ a map such that
(i) $\gamma(\varnothing) = \varnothing$**;**
(ii) $A \subset \gamma(A)$**;**
(iii) $\gamma(\gamma(A)) = \gamma(A)$**;**
(iv) $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$**.**
Then the family $\mathcal{T} = \{X \setminus \gamma(A)\,,\, A \subset X\}$ is a topology in which $\overline{A} = \gamma(A)$.

First, we prove that $A \subset B \Rightarrow \gamma(A) \subset \gamma(B)$. To do so, observe that $A \subset B$ is equivalent to $A \cup B = B$. Therefore, $\gamma(B) = \gamma(A \cup B) \overset{\textbf{(iv)}}{=} \gamma(A) \cup \gamma(B) \supset \gamma(A)$.

(O1) The subset $X = X \setminus \varnothing \overset{\textbf{(i)}}{=} X \setminus \gamma(\varnothing)$ is in $\mathcal{T}$. Moreover, **(ii)** implies that $X = \gamma(X)$ so $\varnothing = X \setminus \gamma(X)$ is also in $\mathcal{T}$.

(O2) Let $\{U_\alpha\}_{\alpha \in J}$ be a family such that $U_\alpha = X \setminus \gamma(A_\alpha)$ for each $\alpha \in J$, and $U = \bigcup_{\alpha \in J} A_\alpha$. De Morgan's Laws imply that

$$X \setminus U = \bigcap_{\alpha \in J} A_\alpha$$

and we want to prove that this set is of the form $\gamma(B)$ for some subset B of X. Since $\bigcap_{\alpha \in J} \gamma(A_\alpha) \subset \gamma(A_\alpha)$ for all $\alpha \in J$, and γ preserves inclusions, we get, for all $\alpha \in J$,

$$\gamma(X \setminus U) \subset \gamma(\gamma(A_\alpha)) \overset{\textbf{(iii)}}{=} \gamma(A_\alpha)$$

so that $\gamma(X \setminus U) \subset \bigcap_{\alpha \in J} \gamma(A_\alpha) = X \setminus U$, the reverse inclusion is guaranteed by **(ii)**, hence $X \setminus U = \gamma(X \setminus U)$, that is,

$$U = X \setminus \gamma(X \setminus U)$$

and $\mathcal{T}$ is stable under arbitrary unions.

(O3) Let $\{U_i = X \setminus \gamma(A_i)\}_{1 \le i \le n}$ be a finite family of elements of $\mathcal{T}$. De Morgan's Laws imply that

$$X \setminus \bigcap_{i=1}^{n} U_i = X \setminus \bigcap_{i=1}^{n} \gamma(A_i) = X \setminus \gamma\left(\bigcap_{i=1}^{n} A_i\right)$$

where the last equality follows from **(iv)** by induction. This shows that $\mathcal{T}$ is stable under finite intersections, which concludes the proof that it is a topology on X.

Let A be a subset of X. Then $\gamma(A)$ is closed by definition of $\mathcal{T}$ and $A \subset \gamma(A)$ by **(ii)** so $\bar{A} \subset \gamma(A)$. Conversely, observe that $X \setminus \bar{A}$, being open, is of the form $X \setminus \gamma(B)$, that is, $\bar{A} = \gamma(B)$ for some $B \subset X$. Since $A \subset \bar{A}$, and γ preserves inclusions, it follows that

$$\gamma(A) \subset \gamma(\bar{A}) = \gamma(\gamma(B)) \overset{\textbf{(iii)}}{=} \gamma(B) = \bar{A},$$

hence $\gamma(A) = \bar{A}$.

(2) (a) **Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)\ ,\ x \in X\}$ is closed in $X \times X$.**

A key observation is that for A and B subsets of X, the condition $A \cap B = \varnothing$ is equivalent to $(A \times B) \cap \Delta = \varnothing$.
Now, assume X Hausdorff and let $(x,y) \in (X \times X) \setminus \Delta$. Since $x \neq y$, there exist disjoint open sets $U_x \ni x$ and $U_y \ni y$. By definition of the product topology, $U = U_x \times U_y$ is a neighborhood of (x,y) and by the preliminary observation, $U \cap \Delta = \varnothing$ so $X \times X \setminus \Delta$ is open hence Δ is closed.
Conversely, assume that Δ is closed and let $x \neq y$ in X. Since (x,y) belongs to $(X \times X) \setminus \Delta$ assumed open, there exists a neighborhood V of (x,y) such that $V \cap \Delta = \varnothing$. Product of open sets form a basis for the topology of $X \times X$, so there exist open sets U_1 and U_2 such that $(x,y) \in U_1 \times U_2 \subset V$ so $(U_1 \times U_2) \cap \Delta = \varnothing$ which, by the preliminary observation again, guarantees that U_1 and U_2 are disjoint neighborhoods of x and y respectively.

(b) **Determine the accumulation points of $A = \left\{\frac{1}{m} + \frac{1}{n}\ ,\ m,n \in \mathbb{Z}_+\right\} \subset \mathbb{R}$.**

Let A' denote the set of accumulation points of A. The fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ implies that $\left\{\frac{1}{p}\ ,\ p \in \mathbb{Z}_+\right\} \cup \{0\} \subset A'$. Let us prove the converse inclusion. First, observe that if an interval (a,b) with $a > 0$ contains infinitely many elements of the form $\frac{1}{m} + \frac{1}{n}$, then one of the variables m and n must take only finitely many values, while the other takes infinitely many values. Now let $x \in A'$ with $x > 0$. For any $\varepsilon > 0$, the set $B_\varepsilon = (x-\varepsilon, x+\varepsilon) \cap A$ must be infinite. Without loss of generality, we can assume that

$$B_\varepsilon = \left\{\frac{1}{m} + \frac{1}{n}\ ,\ m \in F\ ,\ n \in I_m\right\}$$

with F finite and at least one I_m infinite, say I_{m_0}. For all $n \in I_{m_0}$, we have

$$\left|\left|x - \frac{1}{m_0}\right| - \frac{1}{n}\right| \leq \left|x - \frac{1}{m_0} - \frac{1}{n}\right| < \varepsilon.$$

For n large enough, the left-hand side can be made arbitrarily close to $\left|x - \frac{1}{m_0}\right|$, in particular, we get that $\frac{1}{2}\left|x - \frac{1}{m_0}\right| < \varepsilon$. If $x > 0$ is not of the form $\frac{1}{m_0}$ for any $m_0 \in \mathbb{Z}_+$, then there exists a positive minimum value for the numbers $\frac{1}{2}\left|x - \frac{1}{m_0}\right|$ and B_ε cannot be infinite for arbitrarily small values of ε.

(3) **The boundary of a subset A in a topological space X is defined by**

$$\partial A = \bar{A} \cap \overline{X \setminus A}.$$

(a) **Show that $\bar{A} = \mathring{A} \sqcup \partial A$[5].**
If $x \in \mathring{A}$, there exists a neighborhood of A that is included in A. If $x \in \partial A$, in particular $x \in \overline{X \setminus A}$ so every neighborhood of x intersects $X \setminus A$. This is a contradiction so $\mathring{A} \cap \partial A = \varnothing$.
The interior and boundary of A are included in $\bar{A}$ by definition so the inclusion $\bar{A} \supset \mathring{A} \sqcup \partial A$ is trivial. Conversely, let $x \in \bar{A}$. If x has a neighborhood U such that $U \subset A$, then $x \in \mathring{A}$. The alternative is that every neighborhood of x has non-empty intersection with $X \setminus A$, that is $x \in \overline{X \setminus A}$ so that $x \in \partial A$. Therefore, $\bar{A} \subset \mathring{A} \sqcup \partial A$, which concludes the proof.

(b) **Show that $\partial A = \varnothing$ if and only if A is open and closed.**

By definition of the interior and the closure, $\mathring{A} \subseteq A \subseteq \bar{A}$ and A is open and closed if and only if $\mathring{A} = \bar{A}$. By (a), this is equivalent to $\partial A = \varnothing$.

(c) **Show that U is open if and only if $\partial U = \bar{U} \setminus U$.**

The result of (a) states that U and $\mathring{U}$ are complements in $\bar{U}$, so $\mathring{U} = \bar{U} \setminus \partial U$ and U is equal to $\mathring{U}$, that is, U is open if and only if $U = \bar{U} \setminus \partial U$, which is equivalent to the condition $\partial U = \bar{U} \setminus U$.

(d) **If U is open, is it true that $U = \mathring{\bar{U}}$?**

If U is open, the inclusion $U \subset \bar{U}$ implies that $U \subset \mathring{\bar{U}}$. However, the reverse inclusion may fail: consider for instance $U = \mathbb{R} \setminus \{0\}$ in $\mathbb{R}$. It is open as the union of open intervals and $\bar{U} = \mathbb{R}$ so that $\mathring{\bar{U}} = \mathbb{R} \supsetneq U$.

(4) **Find the boundary and interior of each of the following subsets of $\mathbb{R}^2$.**
(a) $A = \{(x, y)\ ,\ y = 0\}$
(b) $B = \{(x, y)\ ,\ x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \mathbb{Q} \times \mathbb{R}$
(e) $E = \{(x, y)\ ,\ 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{(x, y)\ ,\ x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

Note that, except for (d), a picture is very helpful to determine the boundary and interior of the subsets at hand before rigorously justifying the intuition, using what is known about the (metric) topology of $\mathbb{R}^2$.

[5]The *disjoint union* symbol $\sqcup$ is used to indicate that the sets in the union have empty intersection.

(a) Observe that A is closed, as the complement of $\mathbb{R} \times (-\infty, 0) \cup (0, +\infty)$ which is open as a product of open sets. Another way to see this is to remark that every element of $\mathbb{R}^2 \setminus A$ is of the form (x, y) with $y \neq 0$, and for any $x \in \mathbb{R}$, the basis element

$$V = (x-1, x+1) \times \left(y - \frac{|y|}{2}, y + \frac{|y|}{2}\right)$$

satisfies $(x, y) \in V \subset \mathbb{R}^2 \setminus A$.
Moreover, the interior of A is empty: every element of A is of the form $(x, 0)$, any neighbourhood of which contains a basis element $(a, b) \times (c, d)$ with $c < 0 < d$, which in turn cannot be included in A, for it contains $(x, \frac{d}{2}) \notin A$.
We conclude that $\mathring{A} = \varnothing$ and $\partial A = A$.

(b) Note that $B = (0, +\infty) \times (-\infty, 0) \cup (0, +\infty)$ is open as a product of open sets. Another way to see this is to consider $(x, y) \in B$, that is, $x > 0$ and $y \neq 0$. Then

$$V = \left(\frac{x}{2}, \frac{3x}{2}\right) \times \left(y - \frac{|y|}{2}, y + \frac{|y|}{2}\right)$$

is a neighborhood of (x, y) that is contained in B, which is therefore open.
Finally, B is open because it is the inverse image of $\mathbb{R}^2 \setminus A$ open under the continuous map $(x, y) \mapsto (\ln x, y)$.
Let us prove that the closure of B is the closed half-plane R defined by $x \geq 0$. Let V be a neighborhood of $(x, y) \in R$. If $(x, y) \in B$, there is nothing to prove. If $xy = 0$, then V contains a subset of the form $(a, b) \times (c, d)$ with $0 < b$ and $cd \neq 0$ so $\left(\frac{x+b}{2}, \frac{y+d}{2}\right)$ or $\left(\frac{x+b}{2}, \frac{y+c}{2}\right)$ belongs to $V \cap B$, which is therefore not empty. We have proved that $R \subset \bar{B}$. The converse inclusion follows from the same argument invoked to prove that $\mathbb{R}^2 \setminus A$ is open.
Since B is open, it follows from (c) in the previous problem that $\partial B = \bar{B} \setminus B$, that is ∂B is the union of the vertical axis and the positive horizontal axis.

(c) Since $\overline{A \cup B} = \bar{A} \cup \bar{B}$, it follows form (a) and (b) that $\bar{C} = R \cup A$ consists of the points (x, y) such that $x \geq 0$ or $y = 0$.
Next, $\mathring{C}$ is the right half-plane $(0, +\infty) \times \mathbb{R}$: this set is open as the product of open sets and it is maximal. Indeed, if $x \leq 0$, then any neighborhood of (x, y) contains a subset of the form $(a, b) \times (c, d)$ with $a < 0$ and $cd \neq 0$ so $\left(\frac{x+a}{2}, \frac{y+d}{2}\right)$ or $\left(\frac{x+a}{2}, \frac{y+c}{2}\right)$ belongs to $V \cap (\mathbb{R}^2 \setminus C)$, which is therefore not empty.
It follows from the result proved in (a) of the previous problem that $\partial C = \bar{C} \setminus \mathring{C}$ is the union of the vertical axis and the negative horizontal axis.

(d) Every non-empty open interval of $\mathbb{R}$ contains infinitely many rational and irrational numbers, so every product of intervals contains infinitely many elements of D and $\mathbb{R}^2 \setminus D$. Therefore, $\partial D = \mathbb{R}^2$ and, since $\partial D = \bar{D} \setminus \mathring{D}$, it follows immediately that $\mathring{D} = \varnothing$.

(e) First, observe that the set $\Omega = \{(x,y)\ ,\ 0 < x^2 - y^2 < 1\}$ is open, for instance as the inverse image of the open set $(0,1)$ under the map $(x,y) \mapsto x^2 - y^2$, which is polynomial, hence continuous.
A similar argument, shows that $\Gamma = \{(x,y)\ ,\ 0 \leq x^2 - y^2 \leq 1\}$ is closed. Since $\Omega \subset E \subset \Gamma$, we get the chain of inclusions $\Omega \subset \mathring{E} \subset \bar{E} \subset \Gamma$, hence

$$\partial E = \bar{E} \setminus \mathring{E} \subset \Gamma \setminus \Omega.$$

In other words, a boundary point (x,y) of E satisfies either $x^2 = y^2$ or $x^2 - y^2 = 1$. Conversely, assume that $x^2 - y^2 = 1$. Every neighbourhood of (x,y) contains the points $P_\delta = (x+\delta, y)$ for $\delta \in (-\delta_0, \delta_0)$ with $\delta_0 > 0$. Since

$$(x+\delta)^2 - y^2 = 1 + 2\delta(x+\delta),$$

and the quantity $2\delta(x+\delta)$ takes arbitrarily small positive values when δ runs over $(-\delta_0, \delta_0)$, we see that there are points P_δ in $\mathbb{R}^2 \setminus E$ and E so (x,y) is a boundary point of E. One can proceed in the same way to verify that the two lines given by the equation $x^2 = y^2$ are also included in ∂E, which concludes the proof that ∂E consists exactly of the union of the hyperbola with equation $x^2 - y^2 = 1$ and the lines with equations $y = \pm x$.
It also follows that $\mathring{E} = \Omega$. We have already obtained the inclusion $\Omega \subset \mathring{E}$. Conversely, assume that (x,y) is a point in E not in Ω. Then $x^2 - y^2 = 1$ so (x,y) belongs to ∂E which is disjoint from $\mathring{E}$. This proves that $\mathring{E} \subset \Omega$ and the equality.

(f) No new technique is needed to prove that $\mathring{F}$ is the region located strictly below the branches of the hyperbola with equation $xy = 1$, that is,

$$\mathring{F} = \left\{ (x,y)\ ,\ x \neq 0 \text{ and } y < \frac{1}{x} \right\},$$

and that ∂F is the union of the hyperbola and the vertical axis:

$$\mathring{F} = \left\{ (x,y)\ ,\ x = 0 \text{ or } y = \frac{1}{x} \right\}.$$

Problem set 5: continuous maps, the product topology - Elements of solution

(1) (a) **Consider $\mathbb{Z}_+$ equipped with the topology in which open sets are the subsets U such that if n is in U, then any divisor of n belongs to U. Give a necessary and sufficient condition for a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ to be continuous.**

Assume f continuous. For $n \in \mathbb{Z}_+$, let U_n be the open set of all divisors of n. Let $a \in f^{-1}(U_n)$, assumed non-empty. Since f is continuous, $f^{-1}(U_n)$ is open, hence contains all the divisors of a. In other words, if $b|a$, then $f(b) \in U_n$, that is, $f(b)|n$. A necessary condition for continuity is therefore that f preserve divisibility:

$$b|a \Rightarrow f(b)|f(a).$$

Let us prove that the condition is also sufficient. Assume that $f(b)|f(a)$ whenever $b|a$ and let U be open in $\mathbb{Z}_+$. If $f^{-1}(U) =$, it is open. Otherwise, let $a \in f^{-1}(U)$. To prove that $f^{-1}(U)$ is open, it suffices to show that it contains all the divisors of a. The property of f implies that $f(b)|f(a)$ for every such divisor b and, U being open, this implies that $f(b) \in U$, that is, $b \in f^{-1}(U)$.

(b) **Let $\chi_{\mathbb{Q}}$ be the indicator of $\mathbb{Q}$. Prove that the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(x) = x \cdot \chi_{\mathbb{Q}}(x)$ is continuous at exactly one point.**

We shall prove that φ is continuous at 0 and discontinuous everywhere else. Note that $|\varphi(x)| \leq |x|$ for every $x \in \mathbb{R}$. In particular, let $\varepsilon > 0$ and $\delta = \varepsilon$. Then $|x| < \delta$ implies $|\varphi(x)| < \varepsilon$, so φ is continuous at 0.
Now observe that φ is odd and let $a > 0$ be a positive number. Then,

$$\varphi^{-1}(\left(\frac{a}{2}, \frac{3a}{2}\right)) = \left(\frac{a}{2}, \frac{3a}{2}\right) \cap \mathbb{Q}$$

which is not open as no subset of $\mathbb{Q}$ can contain an open interval of $\mathbb{R}$. Therefore, φ is not continuous at a.

(2) **Let X and Y be topological spaces. If A is a subset of either, we denote by A' the sets of accumulation points of A and by ∂A its boundary. Let $f : X \longrightarrow Y$ be a map. Determine the implications between the following statements.**

(i) **f is continuous.**

(ii) **$f(A') \subset (f(A))'$ for any $A \subset X$.**

(iii) **$\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for any $B \subset Y$.**

Considering a constant function $\mathbb{R} \longrightarrow \mathbb{R}$ shows (i)$\not\Rightarrow$(ii). However, the converse is true: let A be a subset of X and $x \in \bar{A} = A \cup A'$.

- If $x \in A$, then $f(x) \in f(A) \subset \overline{f(A)}$.
- If $x \in A'$, then (ii) implies that $f(x) \in (f(A))' \subset \overline{f(A)}$.

Therefore, $f(\bar{A}) \subset \overline{f(A)}$ for any $A \subset X$ so f is continuous by [M. Th. 18.1].

Let us prove that (i) $\Rightarrow$ (iii). Assume f continuous, let $B \subset Y$ be a subset and $x \in \partial(f^{-1}(B))$. If $x \notin f^{-1}(\partial B)$, there are two possibilities.

Case 1: $f(x) \in \mathring{B}$. Then $x \in f^{-1}(\mathring{B})$, open by continuity of f. Since $f^{-1}(\mathring{B}) \subset f^{-1}(B)$, it follows that x is an interior point of $f^{-1}(B)$, which is a contradiction.

Case 2: $f(x) \in Y \setminus \bar{B}$. Then $x \in f^{-1}(Y \setminus \bar{B})$, open by (i). In particular, there is a neighborhood U of x such that $U \subset f^{-1}(Y \setminus \bar{B})$. Since $x \in \partial(f^{-1}(B))$, it follows that there exists some y in U such that $f(y) \in B$, which contradicts the assumption that $f(x) \in Y \setminus \bar{B}$.

Altogether, this proves that $x \in f^{-1}(\partial B)$, hence the inclusion of (iii).

To establish the converse, we rely on the following characterization of continuity:

Lemma: f is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for every $B \subset Y$.

Proof of the lemma: if f is continuous, the inverse image of the closed set $\bar{B}$ is a closed set that contains $f^{-1}(B)$ hence its closure. Conversely, if B is closed, the condition becomes $\overline{f^{-1}(B)} \subset f^{-1}(B)$. The reverse inclusion holds by definition of the closure, so $\overline{f^{-1}(B)} = f^{-1}(B)$, hence $f^{-1}(B)$ is closed and f is continuous.

If f is discontinuous, the lemma implies the existence of some $B \subset Y$ such that $\overline{f^{-1}(B)} \not\subset f^{-1}(\bar{B})$. Let x be an element of $\overline{f^{-1}(B)}$ such that $f(x) \notin \bar{B}$, hence $f(x) \notin B$. Since $\overline{f^{-1}(B)} = f^{-1}(B) \cup \partial f^{-1}(B)$, it follows that $x \in \partial f^{-1}(B)$.

The fact that $f(x) \notin \bar{B}$ implies that $f(x) \notin \bar{B} \setminus \mathring{B} = \partial B$, that is

$$\partial(f^{-1}(B)) \not\subset f^{-1}(\partial B)$$

and we have proved the contrapositive of (iii) $\Rightarrow$ (i).

To sum up, conditions (i) and (iii) are equivalent and they are implied by (ii), but the converse does not hold:

$$\text{(ii)} \Rightarrow \text{(i)} \Leftrightarrow \text{(iii)}.$$

(3) **Let X and Y be topological spaces, and assume Y Hausdorff. Let A be a subset of X and f_1, f_2 continuous maps from the closure $\bar{A}$ to Y. Prove that if f_1 and f_2 restrict to the same function $f : A \to Y$, then $f_1 = f_2$.**

We argue by contradiction: if $f_1 \neq f_2$, there exists $x \in \bar{A}$ such that $f_1(x) \neq f_2(x)$ and $x \notin A$. Since Y is Hausdorff, there exist disjoint neighborhoods V_1 of $f_1(x)$ and V_2 of $f_2(x)$. By continuity of f_1 and f_2, both $f_1^{-1}(V_1)$ and $f_2^{-1}(V_2)$ are neighborhoods of x, and so is $U = f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$. Since $x \in \bar{A} \setminus A$, the neighborhood U contains some a in A such that $f_1(a) = f(a) = f_2(a)$. Therefore, $f(a) \in V_1 \cap V_2$ which is assumed empty.

(4) **Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $X = \prod_{\alpha \in J} X_\alpha$.**

(a) **Give a necessary and sufficient condition for a sequence $\{u_n\}_{n \in \mathbb{Z}_+}$ to converge in X equipped with the product topology.**

Assume that $\lim_{n\to\infty} u_n = l$. The projection maps π_α are continuous so the 'non-necessarily metrizable' part of the sequential characterization theorem [M. Th. 21.3] implies that

$$(\star) \qquad \forall \alpha \in J \quad , \quad \lim_{n\to\infty} u_{n\alpha} = l_\alpha.$$

Conversely, assume that $\lim_{n\to\infty} \pi_\alpha(u_n) = \pi_\alpha(l)$ for every $\alpha \in J$ and let U be a neighborhood of l. We may assume that U is an intersection of cylinders, that is,

$$U = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \ldots \cap \pi_{\alpha_p}^{-1}(U_{\alpha_p})$$

since such elements form a basis for the product topology. With our assumption, there exists, for each $i \in \{1, \ldots, p\}$, a rank N_i such that $\pi_{\alpha_i}(u_n) \in U_{\alpha_i}$ for all $n \geq N_i$. This implies that $u_n \in U$ for all $n \geq \max_{1 \leq i \leq n} N_i$, so $\lim_{n\to\infty} u_n = l$.

(b) **Does the result hold if X is equipped with the box topology?**

The box topology is finer than the product topology so condition $(\star)$ is certainly necessary. It is not sufficient, however, as the following example shows.
Let $J = \mathbb{Z}_+$ and $X_k = \mathbb{R}$ with the standard topology for each $k \in \mathbb{Z}_+$ so that $X = \mathbb{R}^\omega$ as a set. Then, consider the sequence $({}^n u)_{n \in \mathbb{Z}_+}$ defined by

$${}^n u_k = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} .$$

Then $\lim_{n\to\infty} {}^n u_k = \lim_{n\to\infty} \pi_k({}^n u) = 0$ for every $k \in \mathbb{Z}_+$ so ${}^n u$ converges to the zero sequence in the product topology.
On the other hand, the open box $\prod_{k \in \mathbb{Z}_+} (-1, 1)$ is a neighborhood of the zero sequence that contains no term of the sequence $({}^n u)_{n \in \mathbb{Z}_+}$, which therefore cannot converge to zero in the box topology.

Problem set 6: metrizable spaces - Elements of solution

(1) **Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. For $x \in \mathbb{R}^\omega$ and $0 < \varepsilon < 1$, let**

$$P(x, \varepsilon) = \prod_{n \in \mathbb{Z}_+} (x_n - \varepsilon, x_n + \varepsilon).$$

(a) **Compare $P(x, \varepsilon)$ with $B_{\bar{\rho}}(x, \varepsilon)$.**

With $0 < \varepsilon < 1$, the definition of $\bar{\rho}$ implies that

$$B_{\bar{\rho}}(x, \varepsilon) = \left\{ y \in \mathbb{R}^\omega \ , \ \sup_{n \geq 1} |x_n - y_n| < \varepsilon \right\}.$$

On the other hand, $P(x, \varepsilon) = \{y \in \mathbb{R}^\omega \ , \ |x_n - y_n| < \varepsilon \quad \text{for every } n \geq 1\}$ so the inclusion $B_{\bar{\rho}}(x, \varepsilon) \subset P(x, \varepsilon)$ holds by definition of the supremum. This inclusion is strict: the sequence y defined by

$$y_n = x_n + \varepsilon - \frac{1}{n}$$

belongs to $P(x, \varepsilon)$ but $\bar{\rho}(x, y) = \varepsilon$ so $B_{\bar{\rho}}(x, \varepsilon) \subsetneq P(x, \varepsilon)$.

(b) **Is $P(x, \varepsilon)$ open in the uniform topology?**

No: we prove that $P(x, \varepsilon)$ contains no $\bar{\rho}$-ball centered at the element y introduced above. For any $\eta > 0$, the sequence z defined by $z_n = y_n + \frac{\eta}{2}$ satisfies $\bar{\rho}(y, z) < \eta$ hence belongs to $B_{\bar{\rho}}(y, \eta)$.
Observe that $x_n + \varepsilon - y_n < \dfrac{\eta}{2}$ for n large enough, so

$$\begin{aligned} z_n - x_n &= y_n + \frac{\eta}{2} - x_n \\ &= \varepsilon - \frac{1}{n} + \frac{\eta}{2} \\ &> \varepsilon \end{aligned}$$

which means that $z \notin P(x, \varepsilon)$, which is therefore not a neighborhood of y, hence not open.

(c) **Show that $B_{\bar{\rho}}(x, \varepsilon) = \displaystyle\bigcup_{\delta < \varepsilon} P(x, \delta)$.**

Let $y \in P(x, \delta)$ with $\delta < \varepsilon$. Then, $\bar{\rho}(x, y) \leq \delta < \varepsilon$ so $y \in B_{\bar{\rho}}(x, \varepsilon)$, which proves that $B_{\bar{\rho}}(x, \varepsilon) \supset P(x, \delta)$. Since δ was arbitrary, we conclude that

$$B_{\bar{\rho}}(x, \varepsilon) \supset \bigcup_{\delta < \varepsilon} P(x, \delta).$$

Conversely, if $y \in B_{\bar{\rho}}(x, \varepsilon)$, then $y \in P(x, \delta_y)$ with $\delta_y = \bar{\rho}(x, y) < \varepsilon$, so

$$B_{\bar{\rho}}(x, \varepsilon) \subset \bigcup_{\delta < \varepsilon} P(x, \delta).$$

(2) **We denote by $\ell^2(\mathbb{Z}_+)$ the set of square-summable real-valued sequences:**

$$\ell^2(\mathbb{Z}_+) = \left\{ x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega \quad , \quad \sum_{n\geq 1} x_n^2 \quad \textbf{converges} \right\},$$

equipped with the metric

$$d(x,y) = \left(\sum_{n\geq 1} (x_n - y_n)^2 \right)^{1/2}.$$

(a) **Compare the metric topology induced by d on $\ell^2(\mathbb{Z}_+)$ with the restrictions of the box and uniform topologies from $\mathbb{R}^\omega$.**

The inclusion $\mathcal{T}_{\bar\rho} \subset \mathcal{T}_d$ follows from the observation that $B_d(x,r) \subset B_{\bar\rho}(x,r)$ for any $x \in \ell^2(\mathbb{Z}_+)$ and $r > 0$. Indeed, observe that

$$|x_k - y_k|^2 \leq \sum_{n\geq 1} |x_n - y_n|^2$$

for any fixed k, so that $\bar\rho(x,y) \leq d(x,y)$ for any $x, y \in \ell^2(\mathbb{Z}_+)$[6].

This inclusion is strict, as the following example shows. Denote by $\mathbf{0}$ the sequence that is constantly equal to 0. We will prove that $B_d(\mathbf{0},1)$, open in $\mathcal{T}_d$ by definition, is not open in the uniform topology. More precisely, no ball $B_{\bar\rho}(\mathbf{0},r)$ with $r > 0$ is included in $B_d(\mathbf{0},1)$: let $n_0 > \left(\frac{2}{r}\right)^2$ and consider the sequence

$$\xi_n = \begin{cases} \frac{r}{2} & \text{if } n \leq n_0 \\ 0 & \text{if } n > n_0 \end{cases}.$$

Then $\bar\rho(\mathbf{0},\xi) = \frac{r}{2} < r$ so $\xi \in B_{\bar\rho}(\mathbf{0},r)$ but $d(\mathbf{0},\xi) = \sqrt{n_0}\dfrac{r}{2} > 1$ so $\xi \notin B_d(\mathbf{0},1)$.

Next, we prove that $\mathcal{T}_d \subset \mathcal{T}_{\text{box}}$: let $x \in \ell^2(\mathbb{Z}_+)$, $r > 0$ and consider the box

$$\mathfrak{B} = \prod_{n\geq 1} (x_n - \frac{r}{2^n}, x_n + \frac{r}{2^n}).$$

Then $x \in \mathfrak{B} \subset B_d(x,r)$ since $y \in \mathfrak{B}$ implies $d(x,y)^2 \leq \displaystyle\sum_{n\geq 1} \frac{r^2}{4^n} = \frac{r^2}{3}$.

Again, the inclusion is strict: consider the open box

$$\mathfrak{B} = \prod_{n\geq 1} \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Although $\mathbf{0} \in \mathfrak{B}$, no ball $B_d(\mathbf{0},r)$ with $r > 0$ is included in $\mathfrak{B}$.

[6] Another approach to this result consists in proving that uniform convergence implies ℓ^2 convergence and conclude by the sequential characterization of the closure.

Indeed, consider $n_0 > \frac{2}{r}$ and let η be the sequence defined by

$$\eta_n = \begin{cases} \frac{r}{2} & \text{if } n = n_0 \\ 0 & \text{if } n \neq n_0 \end{cases}.$$

Then $x \notin \mathfrak{B}$ but $d(\mathbf{0}, x) = \frac{r}{2} < r$ so $x \in B_d(\mathbf{0}, r)$. Therefore $\mathfrak{B} \notin \mathcal{T}_d$.

We have proved that $\mathcal{T}_{\bar{\rho}} \subsetneq \mathcal{T}_d \subsetneq \mathcal{T}_{\text{box}}$.

(b) **Let $\mathbb{R}^\infty$ denote the subset of $\ell^2(\mathbb{Z}_+)$ consisting of sequences that have finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\ell^2(\mathbb{Z}_+)$.**

We will prove that $\overline{\mathbb{R}^\infty}^{\ell^2(\mathbb{Z}_+)} = \ell^2(\mathbb{Z}_+)$, using the sequential characterization of the closure in a metric space. For any $x \in \ell^2(\mathbb{Z}_+)$, let ${}^n x$ be the sequence defined by

$${}^n x_k = \begin{cases} x_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}.$$

Then, $d({}^n x, x)^2 = \sum_{k>n} x_k^2$. This quantity converges to 0 as $n \to \infty$ because x is square-summable, so $\lim\limits_{n\to\infty} {}^n x = x$ in $\ell^2(\mathbb{Z}_+)$.

(3) **Let X be a topological space, Y a metric space. Assume that $(f_n)_{n\geq 0}$ is a sequence of continuous functions that converges uniformly to $f : X \to Y$.**
Let $(x_n)_{n\geq 0}$ be a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Let $\varepsilon > 0$ and observe that the triangle inequality implies that

$$d(f_n(x_n), f(x)) \leq \underbrace{d(f_n(x_n), f(x_n))}_{A_n} + \underbrace{d(f(x_n), f(x))}_{B_n}.$$

The convergence of the sequence is assumed uniform so there exists an integer N_A such that $d(f_n(\xi), f(\xi)) < \dfrac{\varepsilon}{2}$ for any $n \geq N_A$ and any $\xi \in X$.
In particular, $A_n < \dfrac{\varepsilon}{2}$ for $n \geq N_A$.
Moreover, the Uniform Limit Theorem guarantees that f is continuous. Therefore, there exists N_B such that $B_n < \dfrac{\varepsilon}{2}$ for any $n \geq N_B$.
It follows that $d(f_n(x_n), f(x)) \leq \varepsilon$ for any n greater than $\max(N_A, N_B)$.

(4) **Ultrametric spaces.**

Let X be a set equipped with a map $d : X \times X \longrightarrow \mathbb{R}$ such that

(1) $d(x,y) \geq 0$ (3) $d(x,y) = 0 \Leftrightarrow x = y$

(2) $d(x,y) = d(y,x)$ (4) $d(x,z) \leq \max\left(d(x,y), d(y,z)\right)$

(a) **Verify that d is a distance.**
The only condition to check is the triangle inequality:

$$\begin{aligned} d(x,y) + d(y,z) &= \max\left(d(x,y), d(y,z)\right) + \min\left(d(x,y), d(y,z)\right) \\ &\geq \max\left(d(x,y), d(y,z)\right) && \text{by (1)} \\ &\geq d(x,z) && \text{by (4).} \end{aligned}$$

(b) **Let B be an open ball for d. Prove that $B = B(y,r)$ for every element $y \in B$ for some $r > 0$.**
Let $x \in X$, $r > 0$ and $B = B(x,r)$. Let $y \in B$, that is, assume $d(x,y) < r$. Note that

$$d(x,z) < r \;\Leftrightarrow\; \max(\underbrace{d(x,y)}_{<r}, d(x,z)) < r,$$

which implies that $d(y,z) < r$ by (4). It follows that $B \subset B(y,r)$. The reverse inclusion follows from exchanging x and y in the previous argument. **Every point in the ball is a center!**

(c) **Prove that closed balls are open and open balls are closed in the topology induced by d.**
Let B be a closed ball, that is $B = B_c(x,r) = \{y \in X \ , \ d(x,y) \leq r\}$ for some $x \in X$ and $r > 0$. Let $y \in B$. The open ball $B\left(y, \frac{r}{2}\right)$ is a neighborhood of y contained in B:

$$d(x,z) \underset{(4)}{\leq} \max(\underbrace{d(x,y)}_{\leq r} \ , \ \underbrace{d(y,z)}_{<r/2}) \leq r$$

for any $z \in B\left(y, \frac{r}{2}\right)$, so B is open.

To prove that open balls are closed, we used the sequential characterization: let $(x_n)_{n\geq 1}$ be a sequence in $B(a,r)$ such that $\lim\limits_{n\to\infty} x_n = x$ for some $x \in X$. Since $d(x_n, x)$ can be made arbitrarily small, there exists an integer n_0 such that $d(x_{n_0}, x) < r$. The ultrametric property of d implies that

$$d(a,x) \leq \max(d(a, x_{n_0}) \ , \ d(x_{n_0}, x)) < r,$$

so $x \in B(a,r)$, which is therefore sequentially closed.

Note: it might seem that a distance with such properties may not be useful in any reasonable circumstances or even not exist at all. It is easy to check that the discrete metric on any set is ultrametric. More interestingly the p-adic distances defined on $\mathbb{Q}$ are ultrametric, which gives p-adic analysis a very different flavor from that of real analysis.

Midterm 2: in-class
Elements of solution

Problem 1

1. Show that a topological space is T_1 if and only if for any pair of distinct points, each has a neighborhood that does not contain the other.

Let $x \neq y$ be elements of X, assumed T_1. Then $\{x\}$ is closed so $X \setminus \{x\}$ is a neighborhood of y that does not contain x. Similarly, $X \setminus \{y\}$ is a neighborhood of x that does not contain y. Conversely, assume that distinct points have neighborhoods that does not contain the other and let $x \in X$. Then if $y \neq x$, there is a neighborhood of y that does not contain x so $X \setminus \{x\}$ is open hence $\{x\}$ is closed.

2. Determine the interior and the boundary of the set

$$\Xi = \left\{(x, y) \in \mathbb{R}^2 \ , \ 0 \leq y < x^2 + 1\right\}$$

where $\mathbb{R}^2$ is equipped with its ordinary Euclidean topology.

$$\mathring{\Xi} = \left\{(x, y) \in \mathbb{R}^2 \ , \ 0 < y < x^2 + 1\right\}$$

$$\partial\Xi = \{y = 0\} \cup \{y = x^2 + 1\}$$

Problem 2

Let E be a set with a metric d and $\mathcal{T}_d$ the corresponding metric topology.

1. Prove that the map $d : (E, \mathcal{T}_d) \times (E, \mathcal{T}_d) \longrightarrow \mathbb{R}$ is continuous.

Let (a, b) be an arbitrary basis element for the topology on $\mathbb{R}$, with $b > 0$, so that $d^{-1}((a, b))$ is not empty. Let $(x, y) \in d^{-1}((a, b))$ and $d = d(x, y)$. Then, by the triangle inequality,

$$(p, q) \in B(x, \frac{b-d}{2}) \times B(y, \frac{b-d}{2}) \Rightarrow d(p, q) < b.$$

The triangle inequality also implies that $d(p, q) \geq d(x, y) - d(x, p) - d(v, y)$ so

$$(p, q) \in B(x, \frac{d-a}{2}) \times B(y, \frac{d-a}{2}) \Rightarrow a < d(p, q).$$

It follows that $B(x, r) \times B(y, r)$ with $r = \min\left\{\frac{b-d}{2}, \frac{d-a}{2}\right\}$ is a neighborhood of (x, y) contained in $d^{-1}((a, b))$, which is therefore open.

2. Let $\mathcal{T}$ be a topology on E, such that $d : (E, \mathcal{T}) \times (E, \mathcal{T}) \longrightarrow \mathbb{R}$ is continuous. Prove that $\mathcal{T}$ is finer than $\mathcal{T}_d$.

It suffices to prove that every ball $B(x, r)$ is open for $\mathcal{T}$. If $y \in B(x, r)$, then (x, y) belongs to $d^{-1}((-\infty, r))$, assumed open, so there exists a basis element $U \times V$ in $\mathcal{T} \times \mathcal{T}$ such that

$$(x, y) \in U \times V \subset d^{-1}((-\infty, r)).$$

In particular, V is a neighborhood of y. Moreover, if $z \in V$, then $(x, z) \in U \times V \subset d^{-1}((-\infty, r))$ so $d(x, z) < r$, which proves that $V \subset B(x, r)$, hence $B(x, r) \in \mathcal{T}$.

We have proved that the metric topology is the coarsest topology on E making d continuous.

Problem 3

We prove that the box topology on $\mathbb{R}^\omega$ is not metrizable.

1. Recall the definition of the box topology on $\mathbb{R}^\omega$.

It is the topology generated by the basis $\left\{\prod_{n\geq 1} U_n \text{ , } U_n \text{ open in } \mathbb{R}\right\}$.

Denote by 0 the sequence constantly equal to 0 and let

$$P = (0, +\infty)^\omega = \prod_{n\geq 1}(0, +\infty)$$

be the subset of positive sequences.

2. Verify that 0 belongs to $\bar{P}$.

Let $U = \prod_{n\geq 1} U_n$ be a neighborhood of $\mathbf{0}$. Then U_n is a neigborhood of 0 in $\mathbb{R}$ for every n. Therefore, U_n contains an interval (a_n, b_n) with $a_n < 0 < b_n$ for every n so the sequence $(b_n)_{n\geq 1}$ is an element of $U \cap P$. Every neighborhood of $\mathbf{0}$ meets P so $\mathbf{0} \in \bar{P}$.

3. Prove that no sequence $(p_n)_{n\geq 1} \in P^\omega$ converges to 0 in the box topology.

Let $({}^n u)_{n\geq 1}$ be a sequence of elements of P and consider the open box

$$\mathfrak{B} = \prod_{n\geq 1}(-{}^n u_n, {}^n u_n).$$

Then $\mathbf{0} \in \mathfrak{B}$, but no ${}^n u$ belongs to $\mathfrak{B}$, since the nth term of ${}^n u$ lies outside the nth interval in the product defining $\mathfrak{B}$.

4. Conclude.

In a metrizable space, closure points of are limits of sequences. Here, $\mathbf{0}$ is a closure point of P that is the limit of no sequence of elements of P. Therefore, the box topology on $\mathbb{R}^\omega$ is not metrizable.

Problem 4

1. Let X be a set.

(a) Recall the definition of the uniform topology on $\mathbb{R}^X$.

It is the metric topology associated with $\bar{\rho}(f,g) = \sup_{x\in X} \min\{|f(x)-g(x)|, 1\}$.

(b) Recall the definition of uniform convergence for a sequence in $\mathbb{R}^X$.

The sequence $(f_n)_{n\geq 1}$ converges uniformly to f in $\mathbb{R}^X$ if

$$\forall \varepsilon > 0 \ , \ \exists N_\varepsilon \in \mathbb{Z}_+ \ , \ \forall n \geq N_\varepsilon \ , \ \forall x \in X \ , \ |f_n(x) - f(x)| < \varepsilon.$$

2. Prove that a sequence in $\mathbb{R}^X$ converges uniformly if and only if it converges for $\mathcal{T}_\infty$.

Assume that f_n converges uniformly to f and let $0 < \varepsilon < 1$. Then for $n \geq N_{\frac{\varepsilon}{2}}$ and all $x \in X$,

$$\min\{|f(x) - g(x)|, 1\} = |f_n(x) - f(x)| < \frac{\varepsilon}{2},$$

so

$$\bar{\rho}(f_n, f) = \sup_{x\in X} \min\{|f_n(x) - f(x)|, 1\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

which means that f_n converges to f in the uniform topology.

Conversely, assume that $\lim_{n\to\infty} \bar{\rho}(f_n, f) = 0$ and let $0 < \varepsilon < 1$. For n large enough, $\sup_{x\in X}\{|f_n(x) - f(x)| < \varepsilon$, so that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in X,$$

so f_n converges uniformly to f.

Problem 5

Consider the space $\mathbb{R}^\omega$ of real-valued sequences, with the uniform topology.

1. Prove that the subset B of bounded sequences is closed.

It suffices to prove that a uniform limit of bounded sequences is bounded. Let $({}^n u)_{n\geq 1}$ be such that each $({}^n u_k)_{k\geq 1}$ is bounded:

$$|{}^n u_k| \leq M_n \quad \text{for all } k \geq 1$$

and assume that $\lim_{n\to\infty} {}^n u = u$, uniformly. Then there exists an integer n_0 such that

$$(\star) \qquad \forall n \geq n_0 \ , \ \sup_{k\geq 1} |{}^n u_k - u_k| < 1.$$

If u were unbounded, it would admit a subsequence u_{k_ℓ} such that $\lim_{\ell\to\infty} |u_{k_\ell}| = +\infty$. Since ${}^{n_0}u$ is bounded by M_{n_0}, this would imply that

$$\lim_{\ell\to\infty} |{}^{n_0}u_{k_\ell} - u_{k_\ell}| = +\infty,$$

which contradicts $(\star)$. Therefore, u must be bounded and B contains all its limit points.

2. Let $\mathbb{R}^\infty$ denote the subset of sequences with finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ for the uniform topology.

We will prove that $\overline{\mathbb{R}^\infty} = c_0(\mathbb{Z}_+)$, the set of sequences that converge to 0.

Let $({}^n u)_{n\geq 1}$ be a uniformly convergent sequence of elements of $\mathbb{R}^\infty$ and $u = \lim_{n\to\infty} {}^n u$. If u does not converge to 0, there exists some $\eta > 0$ such that

$$|u_k| > \eta$$

for arbitrarily large values of k. It follows that, for any $n \geq 1$,

$$|{}^n u - u_k| = |u_k| > \eta \quad \text{for some } k$$

since ${}^n u$ has only finitely many non-zero terms. This implies that $\sup_{k\geq 1} |{}^n u_k - u_k| \geq \eta$ for all $n \geq 1$, which contradicts the uniform convergence of $({}^n u)_{n\geq 1}$.

Conversely, any sequence u in $c_0(\mathbb{Z}_+)$ is the uniform limit of its truncations: let ${}^n u$ be the sequence defined by

$${}^n u_k = \begin{cases} u_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} .$$

Then, ${}^n u \in c_0(\mathbb{Z}_+)$ and

$$\sup_{k\geq 1} |{}^n u_k - u_k| = \sup_{k>n} |u_k| \underset{n\to\infty}{\longrightarrow} 0$$

so ${}^n u$ converges uniformly to u.

Midterm 2: take-home
Elements of solution

Problem 1

The purpose of this problem is to study the connected components of $\mathbb{R}^\omega$ in various topologies. In what follows, B and U respectively denote the sets of bounded and unbounded sequences. Note that $\mathbb{R}^\omega = B \sqcup U$.

The sequence whose terms are constantly equal to 0 is denoted by $\mathbf{0}$. The connected component of x is denoted by C_x.

Finally, if x is an element in $\mathbb{R}^\omega$ and A a subset of $\mathbb{R}^\omega$, we denote by $x + A$ the set of A-translates of x, that is

$$x + A = \{x + a \ , \ a \in A\}.$$

1. Determine the connected components of $\mathbb{R}^\omega$ in the product topology.

For x, y in $\mathbb{R}^\omega$, the function

$$s_{x,y} : t \longmapsto (1-t)x + ty$$

is continuous from $[0,1]$ to $\mathbb{R}^\omega$ equipped with the box topology, because each component map s_{x_n,y_n} is polynomial hence continuous from $[0,1]$ to $\mathbb{R}$. Since each $s_{x,y}(t)$ is a real-valued sequence, it follows that $\mathbb{R}^\omega$ is convex, hence (path) connected.

2. Consider $\mathbb{R}^\omega$ equipped with the uniform topology.

(a) **Prove that x is in the same connected component as 0 if and only if x is bounded.**

We know that B is closed in $\mathbb{R}^\omega$ for the uniform topology. A similar argument shows that U is also closed, so that they constitute a separation of $\mathbb{R}^\omega$ in this topology. Therefore, since the zero sequence is bounded, we see that $C_{\mathbf{0}} \subset B$. Furthermore, for $x \in B$, the function $s_{\mathbf{0},x}$ satisfies

$$|s_{\mathbf{0},x}(t)| \leq t \cdot \sup_{n \geq 1} |x_n|$$

for every $t \in [0,1]$. It follows that it is continuous and that every $s_{\mathbf{0},x}(t)$ is bounded so B is connected in the uniform topology. Therefore, $C_{\mathbf{0}} = B$.

(b) **Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the uniform topology.**

For x fixed in $\mathbb{R}^\omega$, the map $y \longmapsto y - x$ is a homeomorphism from $\mathbb{R}^\omega$ onto itself, which sends x to $\mathbf{0}$. It follows that x and y are in the same connected component if and only if $x - y \in C_{\mathbf{0}}$. In other words, $C_x = x + B$.

3. Consider $\mathbb{R}^\omega$ equipped with the box topology.

(a) **Let $x, y \in \mathbb{R}^\omega$ be such that $x - y \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Prove that there exists a homeomorphism $\varphi : \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$ such that $(\varphi(x)_n)_{n\in\mathbb{Z}_+}$ is a bounded sequence and $(\varphi(y)_n)_{n\in\mathbb{Z}_+}$ is unbounded.**

Let us prove the following:
Lemma. Let $(\alpha_n)_{n\in\mathbb{Z}_+}$ and $(\beta_n)_{n\in\mathbb{Z}_+}$ be fixed sequences of real numbers such that $\alpha_n \neq 0$ for all n. Then the map

$$\begin{array}{rccc} \varphi : & \mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & (u_n)_{n\in\mathbb{Z}_+} & \longmapsto & (\alpha_n u_n + \beta_n)_{n\in\mathbb{Z}_+} \end{array}.$$

is a homeomorphism in the box topology.

Proof. Since φ is bijective with inverse $(u_n)_{n\in\mathbb{Z}_+} \longmapsto \left(\frac{u_n}{\alpha_n} - \beta_n\right)_{n\in\mathbb{Z}_+}$ it suffices to prove that every map from $\mathbb{R}^\omega$ to itself with (non-constant) affine components is continuous. Let $W = \prod_{n\geq 1}(s_n, t_n)$ be a basis element for the box topology. Then,

$$\varphi^{-1}(W) = \prod_{n\geq 1}\left(\widetilde{\frac{s_n}{\alpha_n} - \beta_n, \frac{t_n}{\alpha_n} - \beta_n}\right)$$

is also an open box, hence open. Therefore φ is continuous.

Now, with x and y fixed in $\mathbb{R}^\omega$ such that $x - y \notin \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, consider the map φ defined by

$$\varphi(u)_n = \begin{cases} u_n - x_n & \text{if } x_n = y_n \\ e^n \cdot \dfrac{u_n - x_n}{y_n - x_n} & \text{if } x_n \neq y_n \end{cases}$$

for $u \in \mathbb{R}^\omega$. Then φ is a homeomorphism of $\mathbb{R}^\omega$ by the lemma and $\varphi(x) = \mathbf{0} \in B$ while $\varphi(y)$ has an exponentially growing subsequence, hence $\varphi(y) \in U$.

(b) **Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the box topology.**

Let $x, y \in \mathbb{R}^\omega$. Since $B \sqcup U$ is a separation of $\mathbb{R}^\omega$ in the box topology, any homeomorphism φ of $\mathbb{R}^\omega$ must satisfy $\varphi(C_x) \subset B$ or $\varphi(C_x) \subset U$. By the previous question, if follows that $y \in C_x$ implies $x - y \in \mathbb{R}^\infty$. In other words, $C_x \subset x + \mathbb{R}^\infty$.

The converse inclusion follows from a convexity argument. Assume that $x - y \in \mathbb{R}^\infty$. Then, $s_{x,y}(t) = x + t(y - x) \in x + \mathbb{R}^\infty$ for every $t \in [0, 1]$.
Let us prove that this map is continuous between $\mathbb{R}$ and $\mathbb{R}^\omega$ equipped with the box topology. Let $W = \prod_{n\geq 1} I_n$ with I_n an open interval of $\mathbb{R}$ for every $n \in \mathbb{Z}_+$. Then, for every n, the set $J_n = \{t \in [0, 1]\ ,\ x_n + t(y_n - x_n)\}$ is
- empty or equal to $[0, 1]$ if $x_n = y_n$;
- an open interval of $[0, 1]$ if $x_n \neq y_n$.

Since the latter occurs only for finitely many values of n, the inverse image of W under $s_{x,y}$, which is $\bigcap_{n\geq 1} I_n$ is either empty or a finite intersection of open sets, hence open. This proves that $s_{x,y}$ is a continuous path, so that $x + \mathbb{R}^\infty$ is path connected, hence equal to C_x.

Problem 2

Let F be a functor between categories $\mathcal{C}$ and $\mathcal{C}'$. A functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ is said to be a *left adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(X), Y) \cong \mathrm{Hom}_{\mathcal{C}'}(X, F(Y))$$

for all objects $X \in \underline{\mathcal{C}}'$ and $Y \in \underline{\mathcal{C}}$. Similarly, G is called a *right adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{C}'}(F(X), Y)$$

for all objects $X \in \underline{\mathcal{C}}$ and $Y \in \underline{\mathcal{C}}'$.

Recall that the forgetful functor $\mathbb{F} : \mathbf{Top} \longrightarrow \mathbf{Set}$ is defined by

- **$\mathbb{F}((X, \mathcal{T})) = X$ for any set X equipped with a topology $\mathcal{T}$;**
- **$\mathbb{F}(f) = f$ for any continuous map $f : X \longrightarrow Y$.**

If X is a set, let $\mathbb{G}(X)$ denote the topological space obtained by endowing X with the trivial topology $\mathcal{T}_{\text{triv.}} = \{X, \emptyset\}$:

$$\mathbb{G}(X) = (X, \mathcal{T}_{\text{triv.}}).$$

If f is a map between sets, define in addition $\mathbb{G}(f) = f$.

This problem is a reformulation in the language of categories of the following basic properties of the trivial and discrete topologies:

(C1) If $(X, \mathcal{T})$ is a topological space, then any map $(X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_{\text{triv.}})$ is continuous.

(C2) If $(Y, \mathcal{T})$ is a topological space, then any map $(X, \mathcal{T}_{\text{disc.}}) \longrightarrow (Y, \mathcal{T})$ is continuous.

1. Verify that $\mathbb{G}$ is a functor.

At the level of objects, $\mathbb{G}$ sends sets to topological spaces. To verify functoriality, it suffices to check that if f is a morphism in **Set**, then $\mathbb{G}(f)$ is a morphism in **Top** and compatibility with compositions in each categroy. Let X and Y be objects in $\underline{\mathbf{Set}}$, and $f \in \mathrm{Hom}(X, Y)$ that is, f is a map between sets X and Y. Then $\mathbb{G}(f) = f$ by definition and the condition

$$\mathbb{G}(f) \in \mathrm{Hom}(\mathbb{G}(X), \mathbb{G}(Y))$$

is equivalent to f being continuous between $(X, \mathcal{T}_{\text{triv.}})$ and $(Y, \mathcal{T}_{\text{triv.}})$, which follows from (C1). The composition relation is immediate as $\mathbb{G}(gf) = gf = \mathbb{G}(g)\mathbb{G}(f)$. Finally, $\mathbb{G}(\text{Id}_X) = \{x \mapsto x\} = \text{Id}_{\mathbb{G}(X)}$.

2. Prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$.

To prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$, we need to compare

$$\text{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$$

with

$$\text{Hom}_{\mathbf{Set}}(\mathbb{F}((X, \mathcal{T})), Y)$$

for any topological space $(X, \mathcal{T})$ and every set Y. By definition, $\text{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$ is the set of continuous maps from $(X, \mathcal{T})$ to $(Y, \mathcal{T}_{\text{triv.}})$.
On the other hand, $\text{Hom}_{\mathbf{Set}}(\mathbb{F}((X, \mathcal{T})), Y)$ consists of all maps from X to Y. Therefore, it follows from (C1) that every element of $\text{Hom}_{\mathbf{Set}}(\mathbb{F}((X, \mathcal{T})), Y)$ can be seen as an element of $\text{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$. In other words, the natural isomorphism realizing the adjunction is the map

$$\begin{array}{rcl} \text{Hom}_{\mathbf{Set}}(\mathbb{F}((X, \mathcal{T})), Y) & \longrightarrow & \text{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y)) \\ f & \longmapsto & f \end{array}$$

3. Find a left adjoint for $\mathbb{F}$.

Similar arguments and (C2) imply that $\mathbb{H}$ defined on **Set** by $\mathbb{H}(X) = (X, \mathcal{T}_{\text{disc.}})$ and $\mathbb{H}(f) = f$ is a functorial and a left adjoint for $\mathbb{F}$.

Problem set 7: connectedness and compactness - Elements of solution

(1) **Let U be an open connected subspace of $\mathbb{R}^2$ and $a \in U$.**

(a) **Prove that the set Γ_a of points $x \in U$ such that there is a path $\gamma : [0,1] \longrightarrow U$ with $\gamma(0) = a$ and $\gamma(1) = x$ is open and closed in U.**

Let x be an element of Γ_a, connected to a by a path γ, and $r > 0$ such that $B(x,r) \subset U$. Then for $y \in B(x,r)$, the map $\tilde{\gamma} : [0,1] \longrightarrow U$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2(1-t)x + (2t-1)y & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a continuous path joining a to y, so $B(x,r) \subset \Gamma_a$, which is therefore open.

To prove that it is closed, let $x \in U$ be an accumulation point of Γ_a and $r > 0$ such that $B(x,r) \subset U$. Since x is an accumulation point, there exists $y \neq x$ in $B(x,r) \cap \Gamma_a$. Then, as in the proof that Γ_a is open, one can concatenate a path from a to y and the segment from y to x to get a continuous path in U that connects a to x. It follows that $x \in \Gamma_a$, which means that Γ_a contains its accumulation points, hence is closed in U.

(b) **What can you conclude?**

The set of points that can be connected to a by a path is open and closed in U connected. Since it is not empty (it contains a) it is equal to U, which is therefore path connected. In other words connectedness and path connectedness are equivalent for open subsets of $\mathbb{R}^2$.

(2) **Let X be a topological space and $Y \subset X$ a connected subspace.**

(a) **Are $\mathring{Y}$ and ∂Y necessarily connected?**

The answer is negative in both cases. Let $Y_1 = L \cup R \subset \mathbb{R}^2$ be the union of the half plane $L = \{x \leq 0\}$ and the half-cone $R = \{x \geq 0 \ , \ |y| \leq x\}$. Then Y_1 is connected because both L and R are, and they intersect at the origin. On the other hand,

$$\mathring{Y_1} = \{x < 0\} \sqcup \{x > 0 \ , \ |y| < x\}$$

is disconnected, as the two terms in the union are disjoint and open.

To see that a connected set need not have a connected boundary, it suffices to consider a closed interval of $\mathbb{R}$. Another example is that of a closed washer in $\mathbb{R}^2$: let $Y_2 = \{1 \leq x^2 + y^2 \leq 4\}$. It is homeomorphic to the rectangle $[1,2] \times [0, 2\pi)$ *via* polar coordinates, hence connected. However, its boundary consists of two disjoint circles, closed in $\mathbb{R}^2$:

$$\partial Y_2 = \{x^2 + y^2 = 1\} \sqcup \{x^2 + y^2 = 4\}$$

which are closed as inverse images of singletons under a continuous map.

(b) **Does the converse hold?**

No: in $\mathbb{R}$, consider the union of negative real number and positive rationals

$$Y_3 = (-\infty, 0) \cup \mathbb{Q}_+.$$

Then $\mathring{Y}_3 = (-\infty, 0)$ and $\partial Y_3 = [0, +\infty]$ are connected, but Y_3 is disconnected as each rational is alone in its connected component.

(3) **Let (E, d) be a metric space.**

(a) **Prove that every compact subspace of E is closed and bounded.**

In a metric spaces[7], compact sets are closed [M. Th.26.3]. Moreover, assume that K is compact in E and that K can be covered by open balls of radius 1. By compactness, K can be covered by finitely many ball of radius 1, say N. Then the triangle inequality shows that $d(x, y) \leq 2N$, so K is bounded.

(b) **Give an example of metric space in which closed bounded sets are not necessarily compact.**

Let X be an infinite set, equipped with the discrete metric. Then X is closed and bounded for the corresponding metric topology. However, the cover given by all singletons, which are open, since the topology is discrete, does not have any finite subcover.

(4) This problem gives concrete descriptions of the Alexandrov compactifications of some locally compact spaces. The Alexandrov compactification is defined up to homeomorphism and it follows from the universal property that homeomorphic spaces have the same compactification.

(a) **Prove that the Alexandrov compactification of $\mathbb{R}$ is homeomorphic to the unit circle $\mathrm{S}^1 = \{(x, y) \in \mathbb{R}^2 \ , \ x^2 + y^2 = 1\}$.**

Observe that S^1 is compact as a closed and bounded subset of $\mathbb{R}^2$. Moreover, $\mathrm{S}^1 \setminus \{(-1, 0)\}$ can be parametrized by the map $r : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$r(t) = \left(\frac{1 - t^2}{1 + t^2} \, , \, \frac{2t}{1 + t^2} \right).$$

This map is continuous because its components are rational functions with non-vanishing denominators, and takes values in S^1. Its inverse is

$$(x, y) \longmapsto \frac{y}{x + 1},$$

continuous on $\mathrm{S}^1 \setminus \{(-1, 0)\}$ for the same reason.

We have proved that $\mathbb{R} \cong \mathrm{S}^1 \setminus \{(-1, 0)\}$, which proves that $\tilde{\mathbb{R}} \cong \mathrm{S}^1$.

[7] This actually holds for Hausdorff spaces in general.

(b) **Verify that $\mathbb{Z}_+ \subset \mathbb{R}$ is a locally compact Hausdorff space.**

Subspaces inherit the Hausdorff property so $\mathbb{Z}_+$ is Haudorff because $\mathbb{R}$ is. Moreover, every finite subset of $\mathbb{Z}_+$ is compact for the subspace topology, which is discrete, $n \in \mathbb{Z}_+$ can be seen in $\{x\}$, compact and open.

(c) **Prove that the Alexandrov compactification of $\mathbb{Z}_+$ is homeomorphic to $\left\{\frac{1}{n}\ ,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$.**

Again, observe that $\left\{\frac{1}{n}\ ,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$ is closed and bounded in $\mathbb{R}$, hence compact. Moreover, the map $n \longmapsto \frac{1}{n}$ is a homeomorphism between $\mathbb{Z}_+$ and $\left\{\frac{1}{n}\ ,\ n \in \mathbb{Z}_+\right\}$ so $\widetilde{\mathbb{Z}_+} \cong \left\{\frac{1}{n}\ ,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$.

Final examination - Elements of solution

Problem 1

1. Let X be a Hausdorff space and K_1, K_2 disjoint compact subsets of X. Prove that there exist disjoint open sets U_1 and U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.

We know that points can be separated from compact sets in Hausdorff spaces. In other words, for every $x \in K_1$, there exist U_1^x neighborhood of x and U_2^x open containing K_2 such that

$$U_1^x \cap U_2^x = \varnothing.$$

The family $\{U_1^x \ , \ x \in K_1\}$ is an open cover of K_1 compact so we can extract a finite subcover $\{U_1^{x_1} \ \ldots, U_1^{x_n}\}$. Then

$$K_1 \subset \bigcup_{i=1}^{n} U_1^{x_i} \stackrel{\text{def}}{=} U_1,$$

and U_1 is open as the union of open sets. Moreover, $K_2 \subset U_2^{x_i}$ for every $i \in \{1, \ldots, n\}$, so

$$K_2 \subset \bigcap_{i=1}^{n} U_2^{x_i} \stackrel{\text{def}}{=} U_2,$$

open as the finite intersection of open sets. To conclude, observe that U_1 and U_2 are disjoint because $U_1^x \cap U_2^x = \varnothing$ for all x.

2. Let X be a discrete space. Describe the compact subsets of X.

Let K be a compact subset of X. Since the topology is assumed discrete, singletons are open and the family $\{\{x\} \ , \ x \in K\}$ is a covering of K. The fact that a finite subcover can be extracted shows that K must be finite.
The converse holds for non-necessarily discrete topologies so the compact subsets of a discrete space are exactly the finite sets.

Problem 2

A topological space is said *totally disconnected* if its only connected subspaces are singletons.

1. Prove that a discrete space is totally disconnected.

In a discrete space X, singletons are open and closed. Therefore, the connected component of $x \in X$ is $\{x\}$. Another way to see this is to observe that any non-trivial partition of a set is a separation, since all subsets are open and closed in the discrete topology.

2. Does the converse hold?

No, consider the example of $\mathbb{Q}$: any open set of $\mathbb{Q}$ contains a subset of the form $(a, b) \cap \mathbb{Q}$ with $a < b$. Such a set contains infinitely many rationals. In particular, singletons are not open, which means that the topology induced by $\mathbb{R}$ is not discrete.
It is however totally disconnected: for any subset A of $\mathbb{Q}$ containing at least two elements q and r, there exists an irrational z such that $q < z < r$, so that $A \cap (-\infty, z) \sqcup A \cap (z, +\infty)$ is a separation of A.

Problem 3

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces; let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$.

1. In $\prod_{\alpha \in J} X_\alpha$ equipped with the product topology, prove that

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

Let $(x_\alpha)_{\alpha \in J}$ be a closure point of $\prod_\alpha A_\alpha$. And consider, for $\beta \in J$, a neighborhood U_β of x_β. Since the projection maps are continuous, $\pi_\beta^{-1}(U_\beta)$ is open in $\prod_\alpha X_\alpha$, hence a neighborhood of $(x_\alpha)_{\alpha \in J}$ so it contains a point $(y_\alpha)_{\alpha \in J} \in \prod_\alpha A_\alpha$. In particular $y_\beta \in U_\beta \cap A_\beta$ so $x_\beta \in \bar{A}_\beta$.

Conversely, let $(x_\alpha)_{\alpha \in J} \in \prod_\alpha \bar{A}_\alpha$ and $U = \prod_\alpha U_\alpha$ a neighborhood of $(x_\alpha)_{\alpha \in J}$. Then every U_α contains a point $y_\alpha \in U_\alpha \cap A_\alpha$ so $(y_\alpha)_{\alpha \in J} \in U \cap \prod A_\alpha$, which means that $(x_\alpha)_{\alpha \in J} \in \overline{\prod_\alpha A_\alpha}$.

2. Does the result hold if $\prod_{\alpha \in J} X_\alpha$ carries the box topology?

Yes. Observe that the continuity of the projection maps used in the previous question still holds in the box topology. The other part of the proof also carries over without change.

Problem 4

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

Assume the existence of a homeomorphism $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$ and consider the restricted map $\tilde{\varphi} = \varphi\big|_{\mathbb{R} \setminus \{0\}} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{f(0)\}$. Then $\tilde{\varphi}$ is bijective by construction and continuous as the restriction of a continuous function. Observe that $\tilde{\varphi}^{-1}$ is continuous for the same reason, which means that $\tilde{\varphi}$ is a homeomorphism between the disconnected space $\mathbb{R} \setminus \{0\}$, and the connected connected space $\mathbb{R}^2 \setminus \{f(0)\}$, which is impossible since connectedness is a topological property.

Problem 5

Let (E, d) be a metric space. An *isometry of* E is a map $f : E \longrightarrow E$ such that

$$d\left(f(x), f(y)\right) = d(x, y)$$

for all $x, y \in E$.

1. Prove that any isometry is continuous and injective.

Let f be an isometry. Then, injectivity follows from the equivalences

$$f(x_1) = f(x_2) \Leftrightarrow d(f(x_1), f(x_2)) = 0 \Leftrightarrow d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2.$$

To prove that f is continuous, let $(x_n)_{n\geq 1}$ be a sequence that converges to $x \in E$. Then

$$\lim_{n\to\infty} d(f(x_n), f(x)) = \lim_{n\to\infty} d(x_n, x) = 0$$

so $\lim\limits_{n\to\infty} f(x_n) = f(x)$, which proves that f is sequentially continuous at any x in E metric.

Assume from now on that E is compact and f an isometry. We want to prove that f is surjective. Assume to the contrary the existence of $a \notin f(E)$.

2. Prove that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subset E \setminus f(E)$.

It suffices to prove that $f(E)$ is closed, which follows from the fact that $f(E)$ is compact as the continuous image of E compact. Since E is Hausdorff, any compact in E is closed.

3. Consider the sequence defined by $x_1 = a$ and $x_{n+1} = f(x_n)$. Prove that

$$d(x_n, x_m) \geq \varepsilon$$

for $n \neq m$ and derive a contradiction.

Assume without loss of generality that $1 < n < m$. Then by definition of the sequence,

$$d(x_n, x_m) = d(a, x_{m-n}) = d(a, \underbrace{f(x_{m-n-1})}_{\in f(E)}).$$

Since no point in $f(E)$ is at distance less than ε of a, it follows that $d(x_n, x_m) \geq \varepsilon$.
Now since $(x_n)_{n\geq 1}$ is a sequence in E metric and compact, it admits a subsequence $(u_n)_{n\geq 1}$ with $\lim_{n\to\infty} u_n = \ell \in K$. Then for m and n large enough to guarantee that $d(u_n, k) < \frac{\varepsilon}{2}$ and $d(u_m, k) < \frac{\varepsilon}{2}$, the triangle inequality implies that

$$d(u_n, u_m) \leq d(u_n, k) + d(k, u_m) < \varepsilon,$$

which contradicts the property of $(x_n)_{n\geq 1}$ established above.

4. Prove that an isometry of a compact metric space is a homeomorphism.

we prove that any continous bijection from a compact space to a Hausdorff space is a homeomorphism. Let $f : X \longrightarrow Y$ be such a map. To prove that $g = f^{-1}$ is continuous, it suffices to prove that $g^{-1}(C) = f(C)$ is closed for any C closed in X. Closed subsets

of compacts are compact so C is compact, therefore $f(C)$ is compact too since f is continuous. Finally, compact subsets of Hausdorff spaces are closed so $f(C)$ is closed.

Problem 6

Let X be a set, $\mathscr{P}(X)$ the set of subsets of X and $\iota : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ a map satisfying, for all $A, B \subset X$:

$$\begin{array}{ll} (1) & \iota(X) = X \\ (2) & \iota(A) \subset A \\ (3) & \iota \circ \iota(A) = \iota(A) \\ (4) & \iota(A \cap B) = \iota(A) \cap \iota(B). \end{array}$$

1. Check that $A \subset B \Rightarrow \iota(A) \subset \iota(B)$.

Note that $A \subset B \Leftrightarrow A \cap B = A$, in which case (4) implies $\iota(A) = \iota(A) \cap \iota(B) \subset \iota(B)$.

2. Prove that the family $\mathcal{T} = \{\iota(A)\ ,\ A \in \mathscr{P}(X)\}$ is a topology on X.

Condition (1) says that $X \in \mathcal{T}$. Moreover, (2) implies that $\iota(\varnothing) \subset \varnothing$ so that $\varnothing = \iota(\varnothing)$, which means that $\varnothing \in \mathcal{T}$.
To see that $\mathcal{T}$ is stable under finite intersections, it suffices to prove that the intersection of two elements of $\mathcal{T}$ belongs to $\mathcal{T}$, which is guaranteed by (4), and argue by induction.
Finally, we prove that $\mathcal{T}$ is stable under arbitrary unions. Let $\{A_\alpha\}_{\alpha \in J}$ be a family of subsets of X; we want to prove that $\bigcup_{\alpha \in J} \iota(A_\alpha) = \iota(B)$ for some $B \subset X$. Observe that (2) implies that

$$\bigcup_{\alpha \in J} \iota(A_\alpha) \supset \iota\left(\bigcup_{\alpha \in J} \iota(A_\alpha)\right).$$

Moreover, $\iota(A_\alpha) \subset \bigcup_{\alpha \in J} \iota(A_\alpha)$ for every $\alpha \in J$ so the result proved in (1) gives

$$\iota(A_\alpha) \overset{(3)}{=} \iota\left(\iota(A_\alpha)\right) \subset \iota\left(\bigcup_{\alpha \in J} \iota(A_\alpha)\right).$$

This holds for every $\alpha \in J$ so $\bigcup_{\alpha \in J} \iota(A_\alpha) \subset \iota\left(\bigcup_{\alpha \in J} \iota(A_\alpha)\right)$, hence

$$\bigcup_{\alpha \in J} \iota(A_\alpha) = \iota\left(\bigcup_{\alpha \in J} \iota(A_\alpha)\right).$$

3. Prove that, in this topology, $\mathring{A} = \iota(A)$ for all $A \subset X$.

The definition of $\mathcal{T}$ and (2) imply that $\iota(A)$ is open and a subset of A so $\iota(A) \subset \mathring{A}$. Conversely, $\mathring{(}A)$ is open so it must be of the form $\iota(B)$ for some $B \subset X$. Since $\iota(B) = \mathring{A} \subset A$, it follows from (3) and the result of (1) that

$$\mathring{A} = \iota(B) = \iota(\iota(B)) \subset \iota(A)$$

which concludes the proof.