

# ~ SOLUTIONS ~

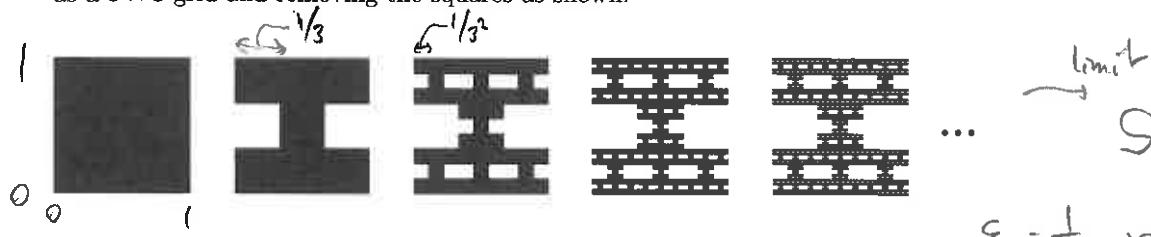
Barnett  
11/10/15

## Math 53: Chaos! 2015: Midterm 2

2 hours, 50 points total, 5 questions, points somewhat  $\propto$  blank space. In each part be sure to explain your answer or give working. Good luck!

1. [5 points]

- (a) Compute the box-counting dimension of the limiting set formed by repeatedly treating each square as a  $3 \times 3$  grid and removing the squares as shown:



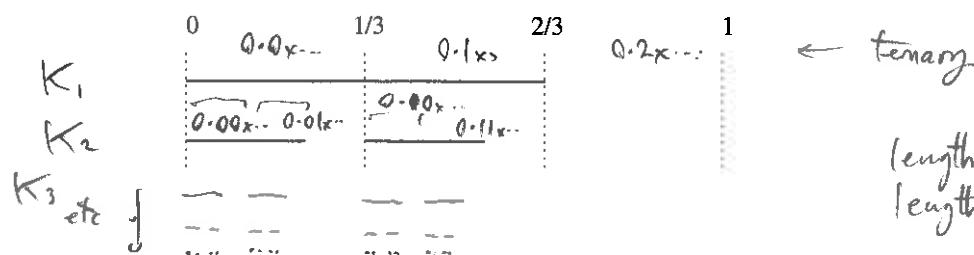
$\varepsilon$	$N(\varepsilon)$
1	1
$1/3$	7
$1/3^2$	$7^2$
$\vdots$	$\vdots$
$1/3^n$	$7^n$

$$\text{boxdim}(S) = \lim_{n \rightarrow \infty} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon_n}$$

$$= \lim_{n \rightarrow \infty} \frac{\ln 7^n}{\ln 3^n} = \frac{\ln 7}{\ln 3}$$

$\varepsilon_n = \frac{1}{3^n}$  valid discrete sequence to use since  $\lim_{n \rightarrow \infty} \frac{\ln \varepsilon_n}{\ln \varepsilon_n} = 1$

2. [15 points] Consider the "last-third Cantor set"  $K$  defined as follows: i) Start with  $[0, 2/3]$ . ii) Split each remaining interval equally into two pieces then remove the last third of each piece. iii) Take the limit of repeating step ii) forever. For clarity, the results of the first two steps are shown here:



$$\text{length}(K_1) = \frac{2}{3}$$

$$\text{length}(K_2) = \left(\frac{2}{3}\right)^2$$

- [2] (a) What numbers expressed in ternary (base 3) are in  $K$ ?

All numbers that can be written using just 0 & 1 in ternary  
 All right endpoints such as  $0.\overline{2} = \frac{2}{3}$ ;  $0.\overline{12} = \frac{5}{9}$ , etc are cut away.  
 eventually: [So no issues with endpoints  $0.\overline{1} = 0.\overline{01}$  as middle-thirds  
 Cantor set.]

[3] (b) Is  $x = 5/26$  in  $K$ ? expand in ternary & if 2 is needed, not in  $K$ .

Eg.  $3k \text{ (mod 1)}$  map:

$$\begin{array}{ccccccc} \frac{5}{26} & \xrightarrow{f} & \frac{15}{26} & \xrightarrow{f} & \frac{45}{26} = \frac{19}{26} & \xrightarrow{f} & \frac{57}{26} = \frac{5}{26} \\ \text{mod 1} & \text{no} & \text{mod 1} & \text{needed "0"} & \text{subtracting 1} & \text{mod 1} & \text{needed "1"} \\ & & & & \text{needed "1"} & & \text{mod 1} \\ & & & & & & \text{mod 1} \Rightarrow "2" \end{array}$$

repeat  $\downarrow \Rightarrow x = 0.\overline{012}$   
needs 2,  
not in K

[4] (c) How many points are in  $K$ : finite, countably infinite, or uncountably infinite? Prove your answer.

Writing each  $x \in K$  in ternary gives  $0.01100010\ldots$  string which can be interpreted as binary.  $\Rightarrow \exists$  1-1 map from  $K$  to  $[0, 1]$ , which is uncountably infinite.

[5] (d) Is the set  $K$  dense in  $[0, 2/3]$ ?

No, since the open interval  $(2/9, 4/3)$  is not in  $K$ , and a dense set cannot avoid hitting every open interval.

[6] (e) Prove whether  $K$  has zero measure in  $\mathbb{R}$  or not.

$K$  can be covered by a countable set of intervals whose total length can be chosen arbitrarily small. Eg to cover with length  $(\frac{2}{3})^n$ , use the intervals  $K_n$  shown in the construction of the set  $K$ . Note  $K_n$  covers  $K$ . For any  $\varepsilon > 0$ , there is an  $n$  such that  $(\frac{2}{3})^n < \varepsilon$ . So  $K$  has zero measure.

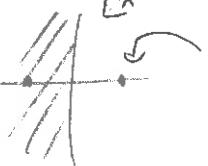
[7] (f) Prove that there's an irrational in  $K$ . [Hint: construction]

$x = 0.1010010001000010\ldots$  in ternary  
is not repeating, thus irrational, and in  $K$ , since "2" is never used.

3. (d) What is the stability of the equilibrium point at  $x = 1$ ? If it is possible to do so, prove your answer.

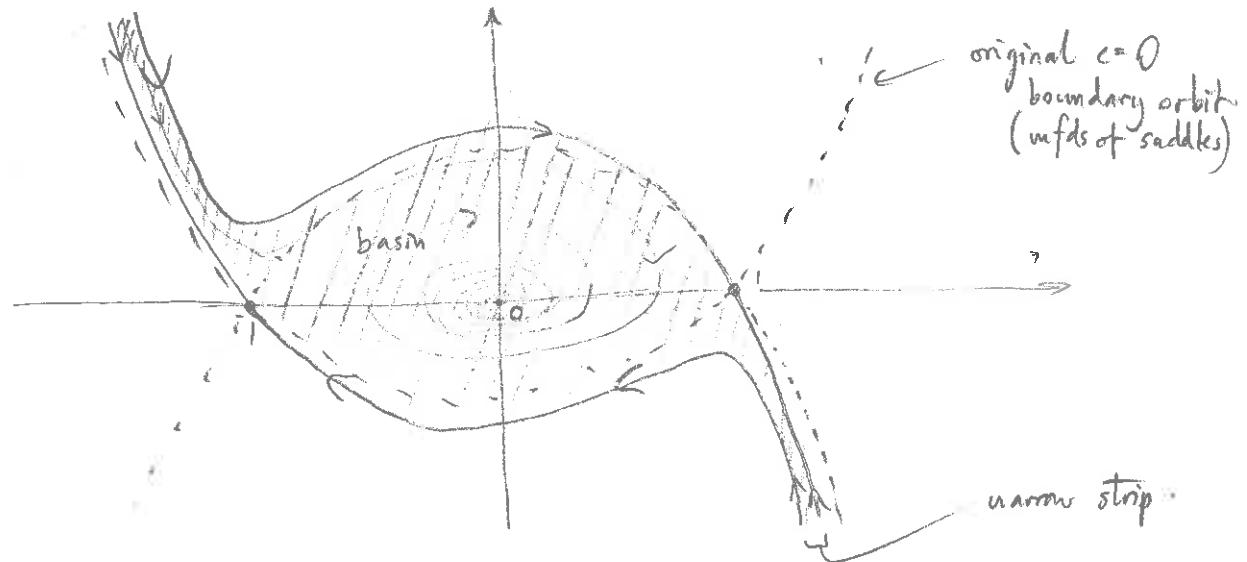
$$\vec{f}(x) = \begin{pmatrix} y \\ -x+x^3 \end{pmatrix} \quad \text{so} \quad \vec{D}\vec{f}(x) = \begin{pmatrix} 0 & 1 \\ -1+3x^2 & 0 \end{pmatrix} \xrightarrow{x=1, y=0} A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

eigenvalues  $\lambda(A) = \pm\sqrt{2}$



there is an eigenvalue with Re part positive,  
= proven unstable

3. (e) Now assume a small amount of damping  $c > 0$  is added. Sketch on a phase plane the *basin* of the equilibrium  $x = 0$ .



2. (f) For damping  $c = 1$ , state and, if possible, prove the stability of the equilibrium  $x = 0$ .

$$\ddot{x} = -\frac{dp}{dx} - c\dot{x} \quad \text{so} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \vec{f}(x) = \begin{pmatrix} y \\ -x+x^3 - cy \end{pmatrix}$$

$$\text{so } \vec{D}\vec{f}(x) = \begin{pmatrix} 0 & 1 \\ -1+3x^2 & -c \end{pmatrix} \xrightarrow{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \text{ evals: } \lambda^2 + c\lambda + 1 = 0$$

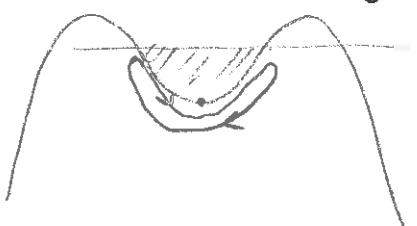
set  $c=1$  as asked

$$\Rightarrow \lambda = -\frac{1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad \text{evals: } \lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

All  $\text{Re } \lambda_j < 0$ , so proven  
Asymptotically stable (AS).

BONUS: What is the full range of periods possible?

(without damping, clearly)



Since the well widens out relative to simple harmonic oscillator  $P(x) = \frac{x^2}{2}$   
(which has period always  $T = 2\pi$ ), we have  
 $2\pi < T < \infty$  ← approached as  $E$  approaches  $\frac{1}{2}V_0$ .

- [2] (g) Describe a *probabilistic game* (set of maps involving a coin toss) that has  $K$  as its attractor.

Toss a fair coin,  $\begin{cases} \text{if Heads, apply } x_{n+1} = \frac{x_n}{3} \\ \text{if Tails, " } x_{n+1} = \frac{x_n + 1}{3} \end{cases}$   
 picture of pdf of  $x$ : 

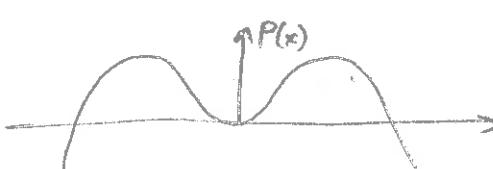
Repeat.

BONUS: what's largest  
 $x \in K? \frac{1}{2} = 0.7$

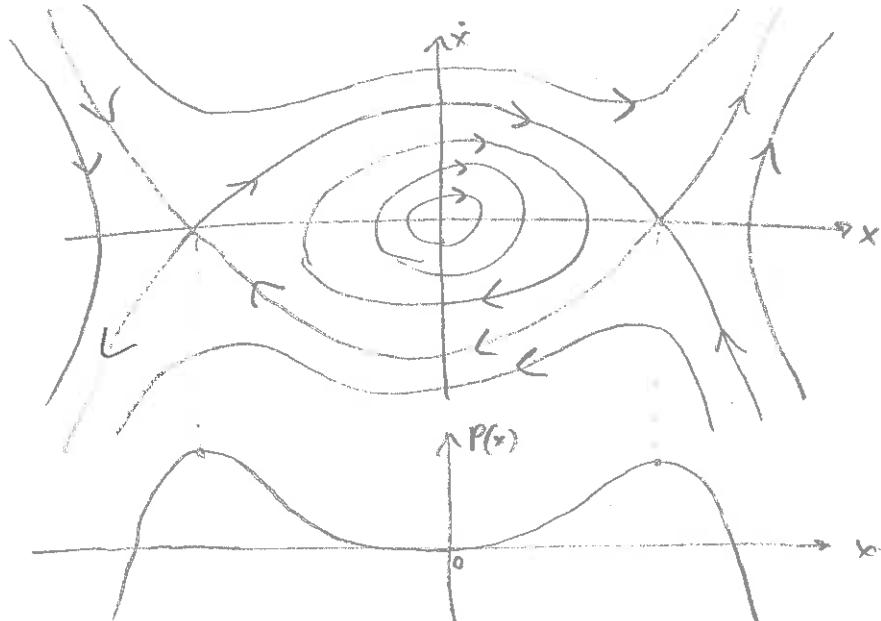
3. [16 points] Consider a particle constrained to move in 1D in the potential  $P(x) = x^2/2 - x^4/4$ .

2. (a) Write a coupled system of two first-order ODEs expressing the motion with no damping.

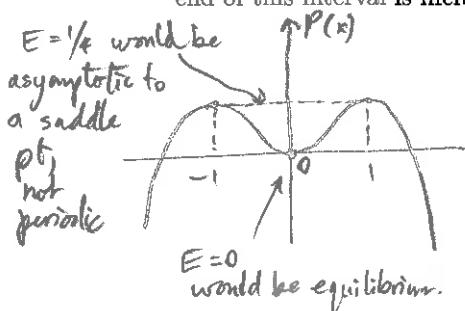
Newton:  $\ddot{x} = -\frac{dP}{dx}$  so  $\dot{x} = y$   
 $\dot{y} = -\frac{dP}{dx} = -x + x^3$



4. (b) Sketch solutions in the phase plane  $(x, \dot{x})$ , showing the full variety of motions that can happen:



2. (c) For what range of total energies can the particle have *periodic* motion? (Be precise whether each end of this interval is included.)



zero force when  $-x + x^3 = 0$ , ie  $x = 0, \pm 1$ .

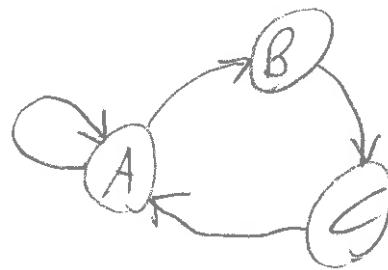
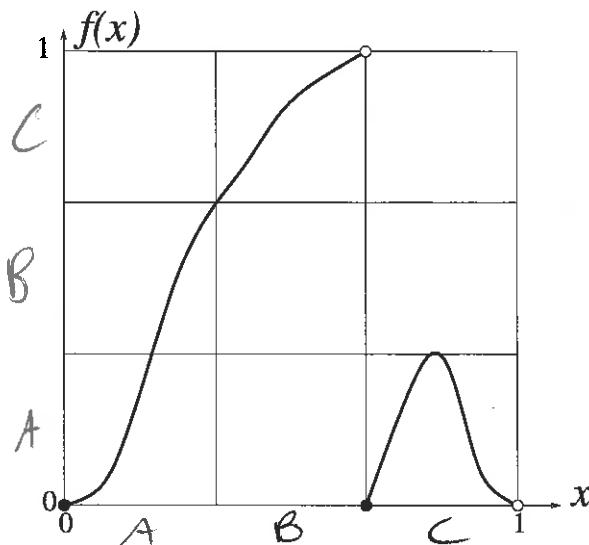
$$P(\pm 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\text{So for } 0 < E < \frac{1}{4}$$

can have periodic motion (could be outside well & not periodic)

needed for Fixed Pt Theorem.

4. [6 points] Consider the following graph of a map  $f$ , which can be viewed as continuous on the periodic interval  $[0, 1]$ .



2. (a) Draw the transition graph for  $f$  in the space above right, using the three intervals shown.  
 4. (b) List all periods of orbits that *must* exist, and prove your answer, mentioning what (if any) theorem(s) you used.

since  $f(A) \supseteq A$ , by Fixed Point Theorem, there  
 is a period-1 orbit (fixed pt of  $f$ ), in  $A$

$f^3(ABCA) = A \supseteq ABCA$  so by FPT  $f^3$  has fp. in  $A$ .  
 This cannot be a lower period since moves through  $A, B, C$   
 $\Rightarrow$  period-3 exists

$f^{3+n}(ABCA^n A) = A$  so period  $3+n$  exists for all  $n \geq 1$ ,  
 not image of lower period because  $ABCA \dots A$   
 doesn't factorize into repeated substrings.

+1 or +2. BONUS: Prove non-existence of any natural numbers missing from this list, taking care with endpoints.

period 2 is missing from list : there is no way to return to starting symbol in transition graph in 2 steps, except  $A \rightarrow A \rightarrow A$ . Here a p-2 is excluded because  $p_1 < p_2 \Rightarrow f(p_1) < f(p_2)$  since  $f$  monotonic increasing in  $A$ , but would need  $f$  to swap  $\{p_1, p_2\}$ .  $A \cup B = \frac{1}{3}$ ,  $B \cup C = \frac{2}{3}$ , but to eliminate period-2 involving such points consider  $\frac{1}{3} \xrightarrow{f} \frac{2}{3} \xrightarrow{f} 0 \xrightarrow{f} 0 \dots$ .

5. [8 points] Unrelated short questions. Provide same amount of explanation as for other questions.

(a) What are the Lyapunov exponents of the cat torus map  $\mathbf{x} = A\mathbf{x} \pmod{1}$  where  $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ ?

$$h_j = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \underbrace{\sqrt{\lambda_j(\mathcal{J}^n \mathcal{J}^{Tn})}}_{= \lambda_j(\mathcal{J})^{2n}} \quad \text{where Jacobian } \mathcal{J} = \vec{DF} = A \text{ wherever defined.}$$

since  $\mathcal{J}$  symmetric.

$$\text{so } h_j = \ln |\lambda_j(A)| : 0 = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 - 1 \quad \text{so } \lambda_j = \frac{3 \pm \sqrt{5}}{2}$$

(b) Is the number  $-2$  in the Mandelbrot set?

$$c = -2$$

$$h_j = \ln \frac{3 \pm \sqrt{5}}{2}$$

If  $z_{n+1} = z_n^2 + c$  (started from  $z_0 = 0$ ) stays bounded,  $c \in M$

Iterate:

$$0 \rightarrow 0^2 - 2 = -2 \rightarrow (-2)^2 - 2 = 2 \rightarrow 2^2 - 2 = 2 \rightarrow 2 \rightarrow 2 \dots$$

eventually fixed, bounded  
⇒ yes. (just!  
It's on boundary)

(c) State the definition of an equilibrium point  $v$  of a flow  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  being (Lyapunov) stable.

$\checkmark$  stable if, for all  $\epsilon > 0$ , there is an open subset  $N_\epsilon \subset N_\epsilon(v)$  with  $v \in N_\epsilon$ , such that for all initial conditions  $\bar{x} \in N_\epsilon$ , the solution  $\vec{F}_t(\bar{x})$  remains in  $N_\epsilon(v)$  for all  $t > 0$ .

(d) Argue whether the set of all periodic points of  $G(x) = 4x(1-x)$  is finite, countably infinite, or uncountably infinite. [Hint: conjugacy]

$G$  is conjugate to tent map  whose iterated graphs are , , etc. There are  $2^k$  fixed points of  $f^k$ , hence at most  $2^k$  periodic orbits of period  $k$ .

These can be listed (e.g. in order of increasing  $k$ ), so are countably infinite. OR, list all L, R strings in order, which count all p.o.s.