Worksheet #6: Stable and unstable manifolds

Let $f(x) = \begin{bmatrix} x/2 \\ 2y - 7x^2 \end{bmatrix}$. $\quad \begin{pmatrix} u \\ v \end{pmatrix} = f(\bar{x}) = \begin{pmatrix} x/2 \\ 2y-7x^2 \end{pmatrix}$ $\quad$ Goal: WRITE x, y in terms of u & V.

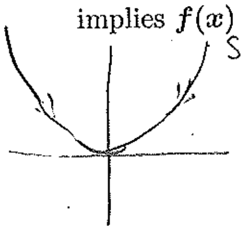
(1) Find an equation for $f^{-1}(x)$.

$U = X/2 \Rightarrow X = 2U$

$V = 2y - 7x^2 \Rightarrow y = \frac{V + 28u^2}{2}$

$\begin{pmatrix} x \\ y \end{pmatrix} = f^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ \frac{V+28U^2}{2} \end{pmatrix}$

(2) Sketch a graph of $S = \{(x, 4x^2) : x \in \mathbb{R}\}$. Show that S is invariant under f (i.e., $x \in S$ implies $f(x)$ and $f^{-1}(x)$ are in S.



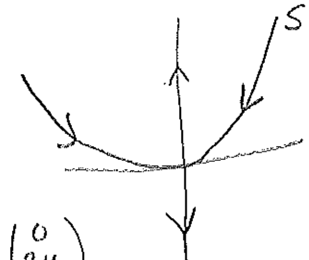
$f\begin{pmatrix} x \\ 4x^2 \end{pmatrix} = \begin{pmatrix} X/2 \\ 2(4x^2) - 7x^2 \end{pmatrix} = \begin{pmatrix} X/2 \\ X^2 \end{pmatrix} = \begin{pmatrix} X/2 \\ 4(X/2)^2 \end{pmatrix} \checkmark \in S$

$f^{-1}\begin{pmatrix} x \\ 4x^2 \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{4x^2 + 28x^2}{2} \end{pmatrix} = \begin{pmatrix} 2x \\ 2x^2 + 14x^2 \end{pmatrix} = \begin{pmatrix} 2x \\ 16x^2 \end{pmatrix} = \begin{pmatrix} 2x \\ 4(2x)^2 \end{pmatrix} \checkmark \in S$

(3) Is S a stable or unstable manifold? Show why this is the case.

Since for any pt $\bar{x} \in S$ $f(\bar{x}) \in S \Rightarrow f^k(\bar{x}) \in S$. S

$\begin{pmatrix} X \\ 4x^2 \end{pmatrix} \xrightarrow{f(\bar{x})} \begin{pmatrix} X/2 \\ x^2 \end{pmatrix} \rightarrow \begin{pmatrix} X/4 \\ x^2/4 \end{pmatrix} \rightarrow$



(4) What is the other manifold? (Hint: fix $x = 0$) $\quad$ if $x = 0$. $f(\bar{x}) = \begin{pmatrix} 0 \\ 2y \end{pmatrix}$

This is the y-axis. This is unstable. Since

$\lim_{n\to\infty} f^{-n}\begin{pmatrix} 0 \\ y \end{pmatrix} = \lim_{n\to\infty}\begin{pmatrix} 0 \\ y/2^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(5) Show that no points outside of S converge to $\mathbf{0}$ under f or f^{-1}.

Pick a pt not on S. $\bar{x} = \begin{pmatrix} x \\ 4x^2 + \varepsilon \end{pmatrix}$ $\quad f(\bar{x}) = \begin{pmatrix} X/2 \\ x^2 + 2\varepsilon \end{pmatrix}$

$f^2(\bar{x}) = \begin{pmatrix} X/4 \\ x^2/4 + 4\varepsilon \end{pmatrix}$ ..

$f^n(\bar{x}) = \begin{pmatrix} X/2^n \\ x^2/2^n + \underbrace{2^n\varepsilon}_{\text{Huge}} \end{pmatrix} \not\to \bar{0}$

likewise for the inverse.