Worksheet #5: 2D linear stability

(1) Consider $\mathsf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Find a condition on the eigenvalues of A such that $\boldsymbol{p} = \mathbf{0}$ is a

(a) sink:

$|a| < 1$ & $|b| < 1$

(b) source:

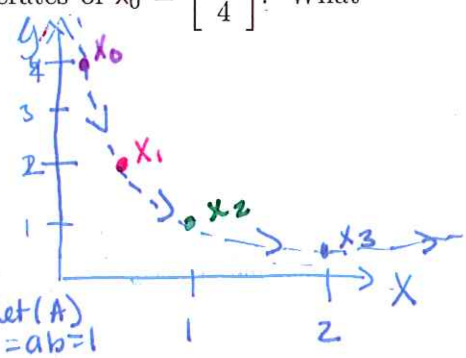
$|a| > 1$ & $|b| > 1$

(c) saddle point

$\left(|a| < 1 \ \& \ |b| > 1 \right)$ OR $\left(|a| > 1 \ \& \ |b| < 1 \right)$

(2) For $\mathsf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ write down and plot the first two iterates of $\mathsf{x}_0 = \begin{bmatrix} \frac{1}{4} \\ 4 \end{bmatrix}$. What curve do they lie on?

$X_0 = \begin{pmatrix} 1/4 \\ 4 \end{pmatrix} \to X_1 = \begin{pmatrix} 1/2 \\ 2 \end{pmatrix} \to X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to X_3 = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$



It is a hyperbola

$xy = 1$ since $x_0 y_0 = 1$ & $\det(A) = ab = 1$

(3) For $\mathsf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, verify that $\mathsf{A}^n = a^{n-1} \begin{bmatrix} a & n \\ 0 & a \end{bmatrix}$.

$n=1$ $\quad A = a^0 \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \Rightarrow$ true for $n=1$

Assume true for $n=k$. Show true for $n=k+1$.

By induction

$$A^{k+1} = A^k A = a^{k-1} \begin{pmatrix} a & k \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = a^{k-1} \begin{pmatrix} a^2 & a + ak \\ 0 & a^2 \end{pmatrix} = a^k \begin{pmatrix} a & k+1 \\ & a \end{pmatrix}$$

$\therefore$ It is true.

(4) Write out $\mathsf{A}^n \boldsymbol{x}$. Use this to decide a condition on a such that the fixed points are a sink or a source.

$$A^n \bar{x} = A^n \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} ax + ny \\ ay \end{pmatrix}$$

if $|a| > 1$ $\lim_{n \to \infty} a^{n-1} \to \infty$. $\Rightarrow$ source.

if $|a| < 1$. We need to worry about $a^{n-1} n$

For simplicity, we will show. $\lim\limits_{n\to\infty} n|a|^{n-1} = 0$.

Let $f(n) = n|a|^{n-1}$

$f(n)$ is a strictly decreasing function for $|a|<1$.

Explanation: $f'(n) = n\,|a|^{n-1}\ln|a| + |a|^{n-1}$

$= |a|^{n-1}(n\ln|a|+1)$ since $|a|<1$, $\ln|a|<0$

< 0 for $n>1$

$\Rightarrow f(n)$ is a decreasing function.

also $f(n)$ has a greatest lower bound of 0.

Thus by the monotonic convergence thm.

$f(n)$ converges as $n\to\infty$. & its limit must be 0.