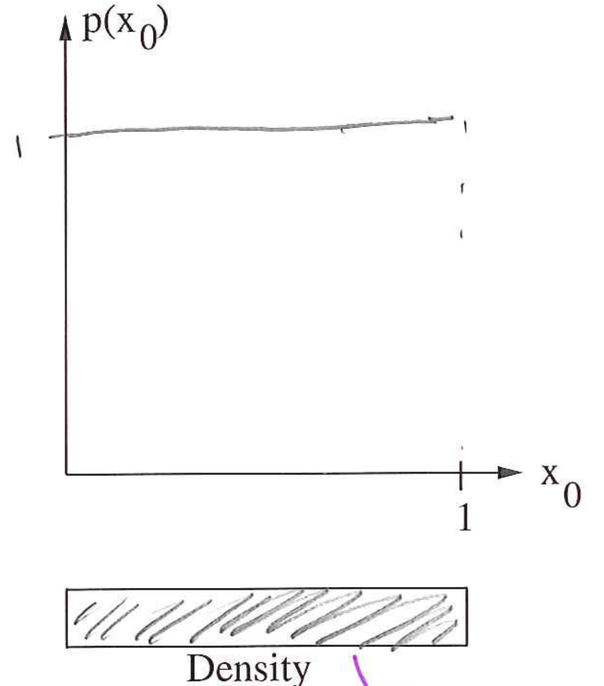
Worksheet #10: Fractals from probablistic games

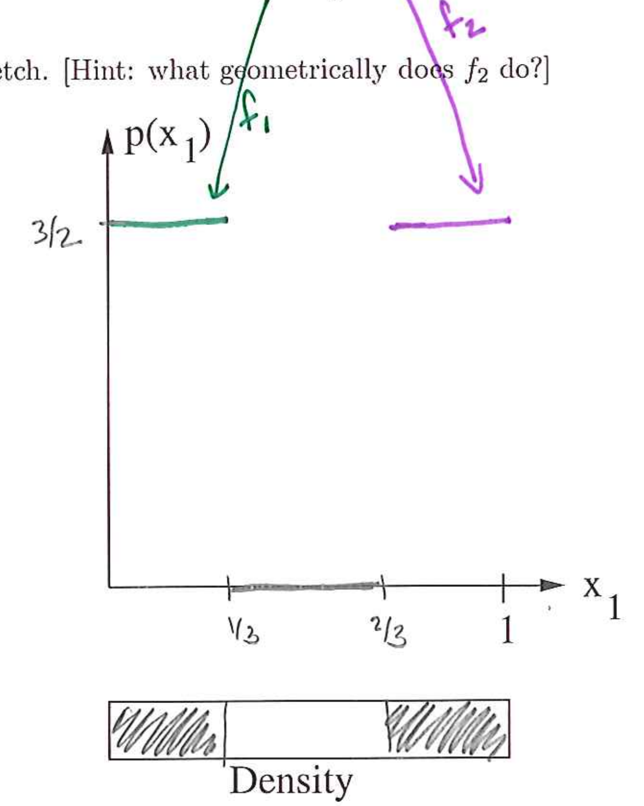
Part 1 Apply $f_1(x) = \frac{x}{3}$ or $f_2(x) = \frac{x+2}{3}$ with equal probability of 1/2 on each iteration.

Starting with $p(x_0)$ uniform on $[0, 1]$,

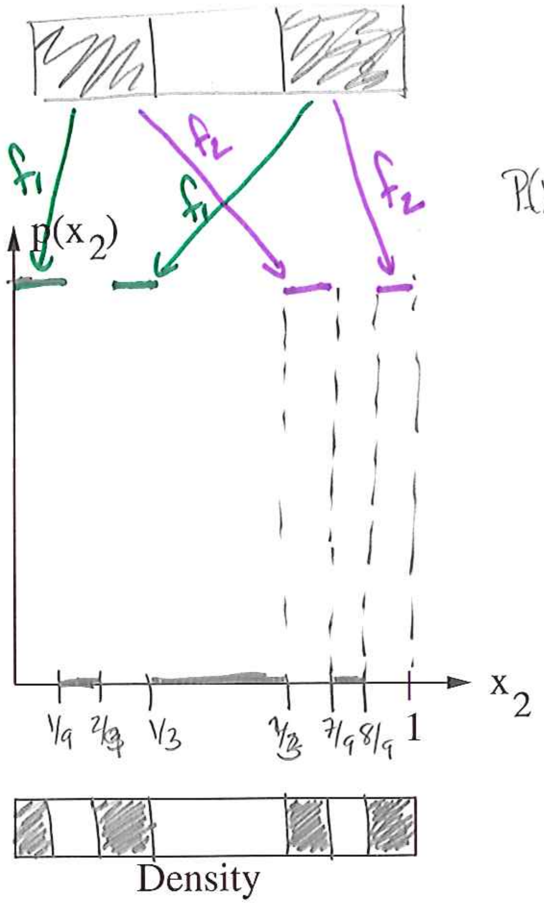


Find $p(x_1)$ and sketch. [Hint: what geometrically does f_2 do?]

$$P(x_1) = \begin{cases} 3/2 & x_1 < 1/3 \text{ or } x_1 > 2/3 \\ 0 & \text{otherwise} \end{cases}$$



Find $p(x_2)$ and sketch.



$$P(X_2) = \begin{cases} 9/4 & \text{if } x < 1/9,\ 2/9 < x < 1/3,\ 2/3 < x < 7/9,\ 8/9 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is $p(x_n)$?

$$P(x_n) = \begin{cases} (3/2)^n & \text{if } x \in K_n \\ 0 & \text{otherwise} \end{cases}$$

What is the limiting attractor set as $n \to \infty$?

K_∞ - the missing third cantor se

Prove an upper bound on the distance of x_n to this set. [Hint: The distance is bounded by the distance from x_0 to the set.]

The maximum distance would be if $x_0 = 1/2$.

$\therefore \text{dist}(x_0, K_\infty) < 1/6$.

upon each iteration you move 3x closer.

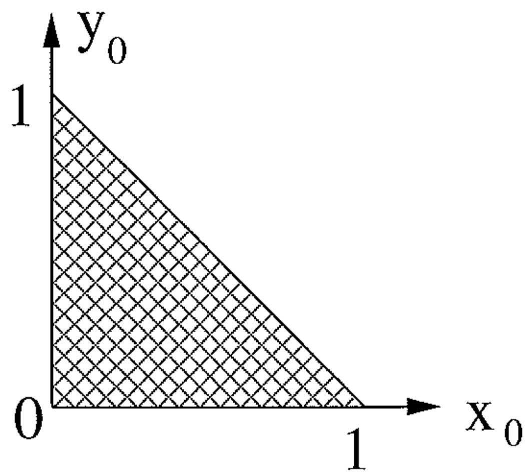
$\Rightarrow \text{dist}(x_n, K_\infty) < \frac{1}{6 \cdot 3^n}$

Part 2 Now try a 2D example. Start with $\boldsymbol{x}_0$ uniform in a triangle.
Apply

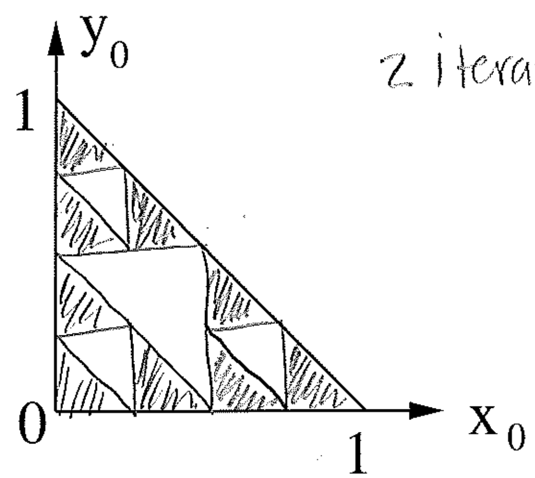
$$\begin{cases} f_1(\boldsymbol{x}) &= \left(\frac{x}{2}, \frac{y}{2}\right) \\ f_2(\boldsymbol{x}) &= \left(\frac{x+1}{2}, \frac{y}{2}\right) \\ f_3(\boldsymbol{x}) &= \left(\frac{x}{2}, \frac{y+1}{2}\right) \end{cases}$$

with probabilities 1/3. Deduce the attractor set.

Uniform



Iterated



2 iterations

Sierpinski gasket.