Math 53 Fall 2013

Chaos!

Second Midterm Exam

Thursday, November 7, 5:00-7:00 PM

Your name (please print): _____

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

_____

For grader use only:

Problem	Points	Score
1	25	
2	20	
3	25	
4	15	
5	15	
Total	100	

1. Consider the second order nonlinear differential equation $x'' + x^2 - 4 = 0$. In this question be careful about *signs*.

   (a) Write the equation as a first order system.

   (b) Find all equilibrium points and use linearization to learn some information about their stability.

   (c) Sketch the phase plane (x, x') showing several orbits including all types of motion that can occur.

(d) What is the potential function $P(x)$ for the differential equation?

(e) In what set of energies do periodic orbits lie? (Watch your endpoints.)

(f) Sketch the phase plane of all points that have an unstable equilibrium as their limit as $t \to \infty$.

2. (a) Find the box-counting dimension of the subset of $\mathbb{R}^2$ formed by the limiting process sketched below: Taking a square partitioning it into 9 equi-sized squares and removing the middle square. In the figure, black boxes are not in the set while white boxes are in the set. [Hint: describe your boxes.]

K_0 K_1 K_2
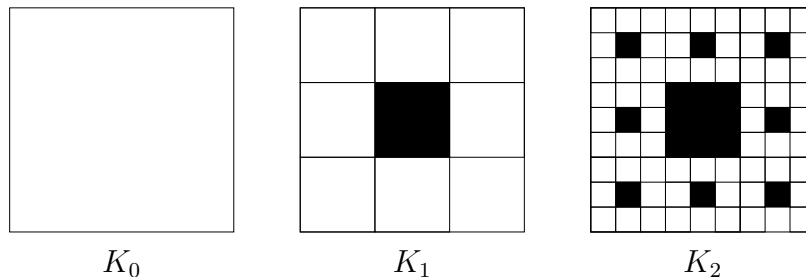
(b) Show that the limiting set has 0 area.

(c) Find the box counting dimension of the set of initial values whose orbits remain bounded for all iterations, under the map

$$T(x) = \begin{cases} 5x & x \leq 1/2 \\ 5(1-x) & x \geq 1/2 \end{cases}$$

[Hint: A picture would be useful. Bonus: what type of set is the attractor?]

(d) What is the measure of the attracting set in (c)? [Bonus: prove it.]

(e) Does the set in (c) contain a countable or uncountable number of points? Finite or infinite?

3. Consider the following map acting on points (x, y) in the unit square $[0, 1]^2$:

$$B(x, y) = \begin{cases} \left(\frac{x}{9}, 2y(\text{mod } 1)\right) & \text{for } y > 1/2 \\ \left(\frac{x+8}{9}, 2y(\text{mod } 1)\right) & \text{for } y \le 1/2 \end{cases}$$

(a) Compute the complete set of Lyapunov exponents (for almost all initial conditions) of this map? Justify your work.

(b) Are most bounded orbits chaotic? Explain your reasoning.

(c) Determine what happens to the area by applying the map by looking at the sum of the Lyapunov exponents.

(d) Assume the map $\boldsymbol{f}$ on $\mathbb{R}^m$ has constant Jacobian $\mathsf{Df}(\boldsymbol{x})$ with with determinant D for each $\boldsymbol{x}$. Prove that the product of all m of the Lyapunov numbers is D (or equivalently, the sum of the Lyapunov exponents is $\ln D$.)

4. Consider the function f with the following graph:
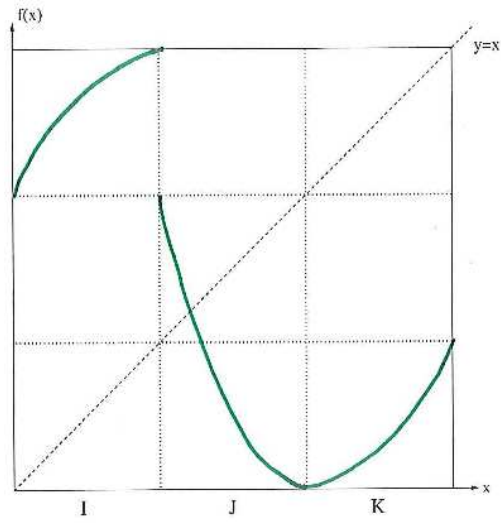
(a) Draw the transition graph.

(b) Give the general statement of the Fixed Point Theorem.

(c) Are there any periodic orbits starting in J? If so, prove their existence.

5. Random short answer questions.

   (a) Is i in the Julia set $J(-i)$? If so, is there something special about i under the mapping $P(z) = z^2 - i$?

   (b) Is $-i$ in the Mandelbrot set?

   (c) Show that the set of odd positive integers is countable?

   (d) Can there exist a conjugacy relation between a set of measure zero and a set with nonzero measure? If so, give an example including the two sets and the conjugacy relation.

BONUS Prove that for the map $P_c(z) = z^2 + c$ with $|c| < 2$, if z_n ever leaves the disc of radius 2 about the origin, the iteration will go to infinity. [Be sure to exclude any finite limits. Hint: use the triangle inequality $|a + b| \leq |a| + |b|$ for $a,\ b \in \mathbb{C}$, but you need to get a lower bound on $|z_{n+1}|$].