Musical variations from a chaotic mapping

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A chaotic mapping provides a technique for generating musical variations of an original work. This technique, based on the sensitivity of chaotic trajectories to initial conditions, produces changes in the pitch sequence of a piece. A sequence of musical pitches $\{p_i\}$, i.e., any piece ranging from Bach (or earlier) to contemporary music, is paired with the x-components $\{x_i\}$ of a Lorenz chaotic trajectory. Each p_i is marked on the x axis at the point designated by its x_i. In this way, the x axis becomes a pitch axis configured according to the notes of the original composition. Then, a second chaotic trajectory, whose initial condition differs from the first, is launched. Its x-components trigger pitches on the pitch axis (via the mapping) that vary in sequence from the original work, thus creating a variation. There are virtually an unlimited number of variations possible, many appealing to expert and nonexpert alike. © *1996 American Institute of Physics.* [S1054-1500(96)00502-7]

I. INTRODUCTION

In recent years, it has been realized that chaos can sometimes be exploited for useful applications. This has been seen in the work of Pecora and Carroll on synchronization of chaotic systems;[1] Cuomo, Oppenheim, and Strogatz on chaotic circuits for private communications;[2] Ditto, Rauseo, and Spano on experimental control of chaos;[3] Bradley on using chaos to broaden the capture range of phase-locked loops;[4] and Roy *et al.* on controlling chaotic lasers.[5] In this paper, chaos is harnessed to yield an application of a rather different sort: The creation of musical variations based on an original piece.

The sensitive dependence property of chaotic trajectories offers a natural mechanism for variability. By affixing the pitch sequence of a musical work to a reference chaotic trajectory, it is possible to generate meaningful variations via a mapping between neighboring chaotic trajectories and the reference. The variations result from changes in the ordering of the pitch sequence. But two chaotic orbits started at nearly the same initial point in state space soon become uncorrelated. To counter this, the mapping was designed so that a nearby trajectory could often track the reference, thus tempering the extent of the separation. Tracking means that pitches in the variation appear exactly where they did in the source. However, regardless of whether the two trajectories track, the mapping links the variation with the original by ensuring only those pitch events found in the source piece comprise the variation.

In this paper, music is used to demonstrate the method, results, and possible applications. The choice of music for illustration is deliberate. It is an application in which context, coherence, and order are paramount. For instance, every pitch in a musical work is a consequence of the pitches that precede it and a foreshadowing of the pitches that follow. The technique's success with a highly context-dependent application such as music, i.e., its ability to generate variations that can be analyzed and used for musical means, indicates it may prove applicable to other sequences of context-dependent symbols, e.g., DNA or protein sequences, pixel sequences from scanned art work, word sequences from prose or poetry, textural sequences requiring some intrinsic variation, and so on.

The variation technique was not designed to alter music of the past. It is meant for music of our own time—for use in the creative process (as an idea generator) and as a springboard for a dynamic music where the written score changes from one hearing to the next. The analyses given in Secs. III A–III E demonstrate how a composer might use the technique as an idea generator, much in the same spirit as composers have taken the inversion*,[6] retrograde*, or retrograde inversion* of a motive, theme, or section, in order to extend their original musical material. Sometimes an inversion is particularly pleasing or stimulating, yet the retrograde turns out blasé. Certainly, musicians are under no obligation to use any of these. This is also true with the variation technique. Any variation can be accepted, altered, or rejected. The artist has choice.

Variations that are close to the original work, diverge from it substantially, or achieve degrees of variability in between these two extremes can be created. Once an entire composition is varied, creating another version of it, the possibility exists for the work to change from one hearing to the next, from one concert to the next, and even within the same concert. The piece is still recognizable as the same piece from concert to concert, but changes have occurred in the score, changes prescribed by the composer. In a broad sense, the music has become dynamic: It changes with time much in the same way a river changes from day to day, season to season, yet is still recognized in its essence.

The application of mathematics to generate or reveal the underlying structure of music has a long history, from the explanation of the overtone series by Pythagoras to the use of numerology by J. S. Bach[7] and the Fibonacci series by Claude Debussy[8] and Béla Bartók.[9] In 1954, Iannis Xenakis proposed a world of sound clouds, masses, and galaxies all governed by new characteristics such as density and rate of change based on probability and stochastic theory.[10] Voss and Clarke claimed in 1978 that the spectral density of fluctuations in the audio power of musical selections ranging from Bach to Scott Joplin, varies as $1/f$ (approximately) down to a frequency of 5×10^{-4} Hz.[11] More recently, statistical methods have been used to analyze J. S. Bach's last fugue, Contrapunctus XIV from *The Art of Fugue*, in order to characterize the data set and postulate a data-driven (where features are learned from the data) approach to its completion.[12]

Fractal and chaotic dynamics have inspired a number of algorithmic approaches to music composition, where the output of a chaotic system is converted into notes, attack envelopes, loudness levels, texture, timbre, and other musical attributes.[13–19] Chaos has also been used to explore sound synthesis, with the intent of creating new instruments and timbres.[20–25] Dynamical system tools such as phase portraits[26] and cusp-catastrophe diagrams[27] have been suggested for analyzing music and explaining paradigm shifts, respectively. Analogies between the language of dynamics and the musical language have been discussed, particularly in reference to whether there is anything inherently musical about the language of nonlinear dynamics and chaos.[28]

While much of the above work with algorithmic composition allows a chaotic system to free-run in order to generate musical ideas, the present work takes a different approach. A given musical piece becomes the source for any number of variations via a chaotic mapping. While these earlier approaches might have some difficulty accommodating disparate musical styles, the technique presented here can take musical sequences of *any* style as input, and produce a virtually infinite set of variations. The stylistic flexibility is encoded in the method by allowing the chaotic mapping to tap the original sequence.

II. THE CHAOTIC MAPPING

Figure 1 illustrates the mapping that creates the variations. First, a chaotic trajectory with an initial condition (IC) of (1,1,1) is simulated using a fourth-order Runge–Kutta implementation of the Lorenz equations,[29]

$$\dot{x}=\sigma(y-x), \tag{1}$$

$$\dot{y}=rx-y-xz, \tag{2}$$

$$\dot{z}=xy-bz, \tag{3}$$

with step size $h=0.01$ and Lorenz parameters $r=28$, $\sigma=10$, and $b=8/3$. This chaotic trajectory serves as the reference trajectory. Let x_i denote the ith x-value in the reference trajectory; the sequence of x-values, obtained after each time step, is plotted in Fig. 1(a). Each x_i is associated with a pitch p_i from the pitch sequence $\{p_i\}$ [Fig. 1(b)] heard in the original work. For example, the first pitch p_1 of the piece is paired with x_1, the first x-value of the reference trajectory; p_2 is paired with x_2, and so on. The pairings continue until every p_i has been given an x_i [Fig. 1(c)]. (Nonmusicians can think of the pitch sequence as a sequence of symbols.)

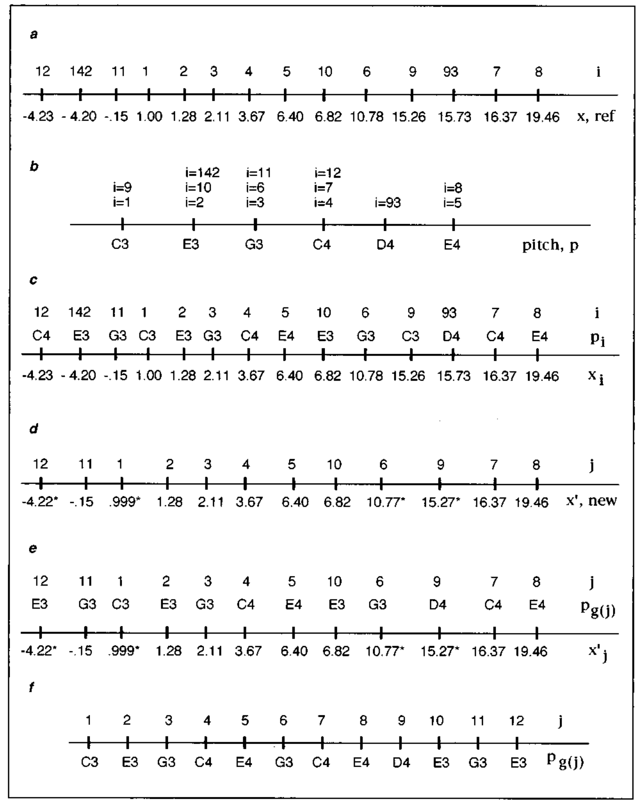

FIG. 1. Generating the first 12 pitches of Variation 1 [Fig. 2(b)]. (a) The first 12 x-components $\{x_i\}$, $i=1,...,12$, of the reference trajectory starting from the IC (1,1,1) are marked below the x axis (not drawn to scale). Two more x-components, that will later prove significant, are indicated: $x_{93}=15.73$ and $x_{142}=-4.20$. (b) The first 12 pitches of the Bach Prelude are marked below the pitch axis. The order in which they are heard is given by the index $i=1,...,12$. The 93rd and 142nd pitches of the original Bach are also given. (c) Parts (a) and (b) combine to give an explicit pairing. (d) The first 12 x'-components of the new trajectory starting from the IC (0.999, 1, 1) are marked below the x' axis (not drawn to scale). Their sequential order is indicated by the index $j=1,...,12$. Those $x'_j \neq x_i$, $i=j$, are starred. (e) For each x-component x'_j, apply the chaotic mapping. All pitches remain unchanged from the original until the ninth pitch. Because $x'_9 = 15.27 \leqslant x_{93} = 15.73$, x'_9 adopts the pitch $D4$ that was initially paired with x_{93}. The next two pitches of Variation 1 replicate the original Bach, but the twelfth pitch, $E3$, arises because $x'_{12} \leqslant x_{142} = -4.20 \mapsto E3$. (f) The variation is heard by playing back $p_{g(j)}$ for $j=1,\ldots,N$, where $N=176$, the number of pitches in the first 11 measures of the Bach.

Next, a new trajectory is started at an IC differing from the reference [Fig. 1(d)]. For each new x-component x'_j, the chaotic mapping is applied,

$$f(x'_j)=p_{g(j)}, \tag{4}$$

where $g(j)$ denotes the index i of the smallest x_i for which $x_i \geqslant x'_j$ [Fig. 1(e)]. In other words, given an x'_j, the smallest x_i is found such that $x_i \geqslant x'_j$. Then its corresponding pitch p_i is assigned to that x'_j. This defines the new pitch $f(x'_j)$. The new variation produced by the chaotic mapping is thus the pitch sequence $p_{g(1)},p_{g(2)},...$, given in Fig. 1(f). Sometimes the new pitch agrees with the original pitch; at other times they differ. This is how a variation can be generated that may retain the flavor of the source.

Note that nearby trajectories do not have to track each other exactly to ensure that many pitches in the variation occur exactly where they did in the source. Rather, the x-components of the two orbits just have to fall within the same region (i.e., pitch dominion) on the x-axis for the two trajectories to effectively track each other in x. This tracking aspect of the chaotic mapping may or may not occur in a variation, depending on ICs, step size, length of the integration, etc. Yet, even when the trajectories do not track, the mapping ensures that only those pitch events occurring in the original sequence will appear in the variation, thus preserving a link between each variation and its source. This linking aspect of the mapping always occurs, regardless of step size, integration length, ICs, etc. The sensitivity of neighboring chaotic trajectories to initial conditions ensures that variability will occur, while the linking and tracking aspects of the chaotic mapping moderate the degree of variation.

The chaotic mapping may implement tracking in x between the reference and new trajectories, resulting in a new pitch agreeing with the original pitch, when $x_i - x_i'$ is greater than zero but sufficiently small. Another case results when $x_i' = x_i$. Then the new pitch must agree with the original pitch (unless an x_i occurred more than once, in which case, the last pitch assigned to the repeated x_i is chosen). Thus, for $x_i - x_i' \geqslant 0$, and the difference sufficiently small, the chaotic mapping can help temper the built-in variability resulting from the sensitive dependence property.

On the other hand, the mapping, in tandem with the sensitivity of chaotic trajectories to initial conditions, is capable of generating a pitch different from the original pitch. Whenever $x_i - x_i' < 0$, the variation will not, in all likelihood, track the source.

A way to examine whether the new and reference trajectories track in x is to plot the difference in x-values, i.e., $x_i - x_i'$, for the duration of the piece. By noting the number of positive, negative, and zero excursions, as well as their magnitudes, one can see if the chaotic mapping has potential for enabling the reference and new trajectories to track in x.

The ability of the mapping to link the variation with the source is maintained regardless of whether the reference and new trajectories have a transient [such as orbits with ICs close to (1,1,1)] or whether the trajectories are (approximately) on the strange attractor. The chaotic mapping guarantees linking because no pitch or chord can occur in the variation that was not already present in the original. The tracking mechanism of the mapping may come into play, whether or not the chaotic trajectories are transient or on the attractor, depending on step size, integration length, ICs, etc.

III. RESULTS AND ANALYSIS

This section uses musical analysis to evaluate several variations generated by the chaotic mapping. More discussion follows in Appendix A. As remarked in an earlier footnote, those musical terms designated by an asterisk are explained in Appendix B.

A. Variations on a prelude by J. S. Bach

To demonstrate the results and determine whether they make musical sense, consider J. S. Bach's Prelude in C Major from *The Well-Tempered Clavier, Book I* (WTC I) as the original work on which two variations are built.[30] A strong harmonic progression*, analogous to an arpeggiated five-part Chorale, underlies the Bach Prelude [Fig. 2(a)]. Variation 1 [Fig. 2(b)] introduces extra melodic elements: the $D4$ appoggiatura*[31] on beat 3 of measure ($m.$) 1; the departure from triadic arpeggios within the first two measures; the introduction of a contrapuntal bass line ($A2$, $B2$, $C3$, $E3$) on the offbeat of $m.$ 5; and the dominant seventh tone on $F4$ heard in $m.$ 7. The above devices were familiar to composers of Bach's time, though they might not have used these melodic elements in quite the same way. (See Appendix A for a discussion and further analysis of Variation 1.)

Variation 2 [Fig. 2(c)] evokes the Prelude, but with some striking digressions; for instance, its key is obscured for the first half of the opening measure. Compared to Variation 1, Variation 2 departs further from the Bach. This is to be expected: The IC that produced Variation 2 is farther from the reference IC than the IC that produced Variation 1.

Like Variation 1, Variation 2 introduces musical elements not present in the source piece, e.g., the melodic turn* [$F4$, ($G3$), $E4$, $F4$, $G4$, ($A3$), $F4$] heard through beats three and four of $m.$ 3. In each of its measures, Variation 2 breaks the pattern of the Prelude, where the second half of each measure replicates the first half, by introducing melodic figuration and superimposed voices. For instance, note the bass motif of $mm.$ 6–8 ($B2$, $B2$, $C3$, $A2$, $D3$, $C3$, $B2$) and the soprano motif of $mm.$ 9–11 ($D4$, $A4$, $C4$, $D4$, $A4$, $G4$, $A4$, $B3$, $E4$, $B3$, $D4$). In the figure, each is indicated by double stems, i.e., two stems that rise (fall) from the note head. (See Appendix A for a discussion and more analysis of Variation 2.)

The original 35-measure Bach Prelude exhibits three prevailing time scales. The slowest is marked by the whole-note because the harmony changes only once per measure. Note that when the pitch sequence changes, the times at which the harmony changes is altered. The fastest time scale is given by the sixteenth-note which arpeggiates or ''samples'' the harmony of the slowest time scale. The half-note time scale represents how often the bass is heard, i.e., the bass enters every half-note until the last three bars ($mm.$ 33–35), when it occurs on the downbeat only. Variation 3 (Fig. 3) alters all three time scales to a greater extent than the previous variations.

The half-note time scale is first disturbed in $m.$ 3, where the bass enters successively on the weakest parts of the sixteenth-note groups, rather than on the much stronger first and third beats of each measure in the original. An example of how the whole-note time scale is broken is given by $m.$ 28, which has the harmonic progression* $I_6 – VII^{4}_{\flat 3}$. The measure possesses two different harmonic chords, rather than the original's one harmony per measure, i.e., the harmonic rhythm* is in half-notes rather than whole-notes. The fastest time scale is disrupted by melodic lines emerging from the

(a)

(c)

(b)

FIG. 2. The pitch sequences of the original Prelude, Variation 1, and Variation 2 (all note durations omitted). The two variations are built upon the first 11 measures of the 35-measure Bach Prelude. The Runge–Kutta solutions for both trajectories encircle the attractor's left lobe eight times and the right lobe three times. The simulations advance 1000 time steps with $h=0.01$. They are sampled every five points ($5=[1000/176]$, where $[\cdot]$ denotes integer truncation and $176=N$). All computations are double precision; the x-values are then rounded to two decimal places before the mapping is applied. Though the differences between graphs of neighboring orbits may not be detectable to the eye, they are to the ear. (a) The first two phrases of Bach's Prelude in *C* Major from the *Well-Tempered Clavier, Book I.* (b) Variation 1, built from chaotic trajectories with new IC (0.999, 1, 1) and reference IC (1,1,1). The chaotic mapping enabled the reference and new trajectories to track in x for 145 out of 176 x-values, resulting in 145 pitches of the variation occurring exactly where they did in the original. (c) Variation 2, built from chaotic trajectories with new IC (1.01, 1, 1) and reference IC (1,1,1). The chaotic trajectories were able to track in x for 98 of 176 x-values, so that 98 pitches in the variation are heard precisely where they occurred in the original.

sixteenth-note motion. They interfere with the sixteenth-note time scale because, as melodies, they possess a rhythm (or time scale) of their own. An example is indicated by slurs in *mm*. 4–6.

Of course, things do not always go perfectly when making these variations. For example, Variation 3 indicates what can occur if an x'_j exists for which there is no $x_i \geqslant x'_j$. Specifically, x'_{342} through x'_{350} of Variation 3 (*m*. 22) exceeded all $\{x_i\}$, resulting in no pitch assignment for these x-values. This is not a problem. When such instances occur, pitches can be inserted by the musician to preserve musical continuity, or the pitches of the original piece can be substituted.

The last pitch event of the Bach Prelude is a five-note *C* major chord at $N=545$. All or part of this chord could be associated with x_N. In general, any musical work that contains pitches simultaneously struck together can generate variations via a pairing that associates any or all of the chord with one or more x_i. However, in this paper, each chord is considered an indissoluble musical event occurring at a specific i in the sequence of N events. So, for the variations discussed here, every pitch or chord event is paired, in sequential order, with its corresponding x_i. Those chords appearing in the variation assume the dynamics (i.e., the loudness levels of the component notes) that each possessed in the original. If a single note appears in the variation, substituting for a chord, it adopts the dynamic level of the lowest

FIG. 3. The pitch sequence of Variation 3, with durations omitted. The mapping was applied to all $N=545$ pitch and chord events of the Prelude, with trajectories having reference IC (1,1,1) and new IC (1, 0.9999, 0.9). The Runge–Kutta solutions for both trajectories encircle the attractor's right lobe once. The simulations advance 545 time steps with $h=0.001$, and are sampled every step. All computations are double precision, with x-values rounded to six decimal places before the mapping is applied. The chaotic mapping fostered tracking in x for 41 out of 545 x-values, resulting in 41 pitch events appearing in the variation exactly where they occurred in the original.

note in the replaced chord. However, if any dynamic level occurs in the variation that is not desirable, the musician can adjust accordingly. Otherwise, the tempo, rhythm, and dynamic levels heard in the variation are the same as the original.

B. Variation 3 as an idea generator

As stated in the Introduction, the variations produced by the chaotic mapping often suggest musical material that can be further developed by a composer. For example, five musical ideas are introduced by Variation 3:

(1) The ''advance'' of the bass. By often appearing a sixteenth-note early, on the weakest beats of the measure, the bass acquires an upbeat quality. Four examples of this are apparent in *mm*. 1 and 2.
(2) Superimposed lines or motives, e.g., the bass melody of *mm*. 4–6 mentioned previously.
(3) Repeated notes. Pairs of repeated notes are heard throughout the variation, starting with $G3$s and $G2$s in *mm*. 11 and 12.
(4) Harmonic sequences*. Measures 12–15 vary the half-sequence* of the original and imply the following harmonic sequence: $VII_3^4/II - II_6 - VII_{\flat 3}^{\,4} - I_6$. But the $E\flat 2 - G1 - E2$ in the fourth beat of *m*. 14 and the $F\sharp 3$, $A2$, and $F3$ of *m*. 15 are extraneous to the harmonic progression. Measures 28 and 29 also suggest a harmonic sequence with the progression $I_6 - VII_{\flat 3}^{\,4} - II_6 - VII_3^4/II$, provided a $B\flat$ is added in the second half of *m*. 29. This sequence is the retrograde of the harmony in *mm*. 12–15. The retrograde occurred in the variation because the new trajectory returned to those regions of the x axis which harbored the original Prelude's half-sequence of *mm*. 12–15 (but from the opposite direction).
(5) The BACH motif. A transposition $(C,B,D,C\sharp)$ of the notes $B\flat,A,C,B\natural$—the musical spelling of Bach's name—appear in the soprano voice of *mm*. 28 and 29. The retrograde of the BACH motif occurs in *mm*. 12–15 of both Variation 3 and the original Prelude.

The five ideas presented by Variation 3 invite development, but not all of the alterations from the original that comprise the variation are desirable. The musician can intervene by rewriting any part of the variation. Taking the musical ideas suggested by the variation technique, the composer follows through on them. They have consequences for the rest of the piece. The good ideas suggest elaboration, and it is here that the composer's art comes into focus.

For example, the inferred sequence of *mm*. 12–15 in Variation 3 can be developed so that its two phrases are each set up with an upbeat in the bass, thus contributing to their parallel structure [Fig. 4(a)]. The $E\flat 2 - G1 - E2$ and $F\sharp 3 - A2 - F3$, extraneous to the harmony in *mm*. 14 and 15 of

FIG. 4. (a) One possible realization of the implied harmonic sequence given by mm. 12–15 (Variation 3). (b) An exact half-sequence built on the inferred harmonic sequence of mm. 28–29 (Variation 3). Both examples illustrate how a composer might develop the suggestions of Variation 3.

FIG. 5. The first 14 measures of Gershwin's First Prelude from *Three Preludes for Piano.*

Variation 3, are eliminated. Instead, the bass of m. 14 is a stepwise transposition of that found in m. 12, and the bass of m. 15 references the bass of m. 13 without transposing it exactly. In order to build what is almost a true half-sequence, the second phrase systematically transposes the harmonic, melodic, and rhythmic patterns of the first phrase until the second beat of m. 15 when the melodic pattern is disrupted in order to intensify the repeated note pair (Idea 3). Measures 28 and 29 can be rewritten as an exact half-sequence [Fig. 4(b)].

C. A Gershwin variation as an idea generator

George Gershwin wrote a set of three preludes for piano, published in 1927. Figure 5 gives the opening page of the First Prelude. A variation of the entire Prelude is shown in Fig. 6. As with the Bach variations, every chord and note in the original Gershwin is treated as a separate pitch event and piggybacked onto the reference trajectory; rhythm, dynamic levels, and tempo are treated as discussed in Sec. III A, last paragraph.

Several ideas emerge from the technique-generated variation, five of which are heard within the first eight bars:

(1) the descending fourth on the 2nd beat of m. 1;
(2) the descending octave, followed by the ascending diminished fifth, of m. 2;
(3) alteration of the four-measure vamp* so that the original m. 3 does not repeat itself;
(4) the melodic answer of m. 8 to m. 7;
(5) additional lower neighbor notes, e.g., the $E\natural$ in beat 2 of m. 8 (lower neighbor to the $F2$).

As with the Bach variations, a musician could take these five ideas and develop them further. Figure 7 gives a short example of how a composer might take three of the five ideas given above and include them in the fortissimo return of the Prelude's main theme (mm. 50–53). Idea 2, the descending octave followed by the ascending diminished fifth, is recalled in m. 51, with the diminished fifth interrupted. An $E\natural$ is added to the bass in the second beat of the same measure (Idea 5). The ''answer'' of m. 8 to the statement of m. 7 (Idea 4) is referenced in m. 53.

D. Variations on additional musical compositions

The design that implements the variation technique has been applied to other works by Bach, Beethoven, Chopin, and Bartók. The point of doing so was to show that one design could accommodate a number of pieces spanning the major styles of Western music from 1700 into the twentieth century. In a series of concert/lectures given in Hong Kong, Chicago, New York, and Boston, it became clear that the musicians in the audiences never agreed on which of these variations were most musical. Two concert pianists who specialize in Bach said the Bach variations were their favorites. Yet a principal percussionist of the Boston Symphony Orchestra disliked the Bach variations, and advised that the Gershwin variation should serve as the best example of this technique, since ''it outdid the original.'' Other professional performers and composers chose variations on a Chopin étude (f minor, Op. 10) and the first movement of a Beethoven sonata (F major, Op. 10) as most relevant to their musical view. Yet to some avid music lovers with no professional training, these variations were least engaging.

E. The variation technique applied to contemporary music

Variations to a Theme (1995, D. S. Dabby) was written so that the variation technique could be applied to a contem-

FIG. 6. A variation of Gershwin's First Prelude. The mapping was applied to all $N=435$ pitch and chord events of the Prelude, with trajectories having reference IC (1,1,1) and new IC (1.0005, 1, 1). The methods are the same as in Fig. 2, except that the simulations are sampled every two points $(2=[1000/435]$, where $[\cdot]$ denotes integer truncation). The chaotic trajectories tracked in x for 361 of 435 events, resulting in parts of the original pitch event sequence appearing unchanged in the variation.

FIG. 7. Measures 50–53 can include three ideas presented by the Gershwin variation of Fig. 6. A composer could similarly develop other parts of the variation to create a composed variation.

porary work. The originating vision for the technique was to use it in tandem with a piece that was considered finished—a piece that made a statement and, hopefully, a compelling one. The composer could then go on a journey with the musical score, in that the variation technique might take the artist elsewhere, to some place new or unimagined.

Scored for piano, *Variations to a Theme* begins out of nowhere, with only subtle references to the Theme. Gradually it telescopes to the Bach Prelude in C (WTC I). This is the Theme, not heard until the very end of the piece, upon which the whole score is based. The work is 12 min long, and, during that time, various references are made to the Bach in its inversion, retrograde, and retrograde inversion (IRRI). These are varied so that they never occur in their original IRRI form. Every measure of the Bach appears in the piece, in at least one of its varied IRRI forms.

After the score was composed, the variation technique was applied to it. One such application of the technique led to the concluding Theme given in Fig. 8. This composed Theme was based on six ideas generated by the technique that appeared in a variation of the Cadenza, Chorale, and Prelude which conclude *Variations to a Theme*. Nested within the composed Theme are a number of motifs that occurred previously in the original work. It was also influenced by Variation 3 (Fig. 3).

For concert performances of *Variations to a Theme*, the composer has written several variations of the concluding Prelude (Theme), all based on variations generated by the chaotic mapping. The musical score may end with the original Bach Prelude [Fig. 2(a)], a composed variation based on Variation 3 (Fig. 3) of the Bach Prelude, or the composed variation of the Prelude (Fig. 8). The written score of the concluding Theme has become dynamic, i.e., it can change with successive hearings. More generally, an entire composition can be varied, creating another version of it, so that the whole score, though recognized in its essence, has become dynamic and self-referencing.

IV. REMARKS

By extending the mapping to the y,z axes, variations can be generated that differ in rhythm (duration) and dynamic (loudness) level, as well as pitch.[28] Here, as elsewhere, it is important to keep in mind that any duration, dynamic level, or pitch event that does not agree with the musician's sensibility can be changed.

Factors affecting the nature and extent of variation are choice of the IC, step size, length of the integration, and the amount of truncation and round-off applied to the trajectories. For instance, if the step size is too big (e.g., $h=0.1$), the x-values quickly separate from one another, effectively eliminating the potential tracking ability of the chaotic mapping, though the linking aspect of the mapping will still hold. If the step size is small ($h=0.001$), the reference and new trajectories may be less likely to track in x due to little space between neighboring x-values on the x axis, especially if the phase portrait encircles the lobes of the attractor several times.

When initial conditions are chosen far apart for the reference and new trajectories, the variations sound like they have diverged considerably from the original. How acceptable this divergence is depends on the individual piece. Assuming the same methods of Fig. 2, new trajectory IC (2,2,3) and reference trajectory IC (1,1,1) resulted in a Bach Prelude variation that was listenable, though not analyzable harmonically, while new IC (100, 0.7, 87) vs. reference IC (1,1,1) generated a Gershwin variation which caught several listeners' ears (assuming same methods as Fig. 6). However, there are other instances where large disparities in ICs between reference and new trajectories produce undesirable results. For example, using the methods of Fig. 3, a new IC of $(-10, -10, -10)$ and reference IC (1,1,1) produced a Bach variation which commenced with 235 soundings of the last C major chord of the original, before proceeding to more interesting territory. In this case, the most negative value for all x_i was $x_{545}=-3.362\,097$, which was paired with the C major chord. However, since the step size was small ($h=0.001$) and $x'_1 = -10$, it took 235 time steps for the new trajectory x-values to reach a number that was greater than $-3.362\,097$. Until an x'_j became greater than x_{545}, the chaotic mapping assigned C major chords to the variation. At $x'_{236}=-3.312\,165$, a $D3$ was mapped to the variation, thus ending the repeated chords.

Though the Lorenz system can exhibit periodic behavior, the mapping is most effective with chaotic trajectories. This is due to their infinite length, enabling sequences of any duration to be piggybacked onto them, and their extreme sensitivity to the IC. To see the drawback of limit cycle behavior, the same methods discussed in Fig. 2 were applied to orbits near the limit cycle for $r=350$.[32] The IC $(-8.032\,932, 44.000\,195, 330.336\,014)$ is on the cycle (approximately). In this case, however, if a trajectory starting at that IC serves as the reference for the mapping, a new trajectory, with its IC obtained by truncating the last digit of the reference IC, yields the original Prelude. That is, the IC $(-8.032\,93, 44.000\,19, 330.336\,01)$ does not give a variation.

FIG. 8. A composed Theme for *Variations to a Theme* (D. S. Dabby, 1995), which resulted from a technique-generated variation of the last three sections of the piece.

Variations 1 and 2 resulted from new and reference trajectories whose x-values were rounded to two decimal places before being paired with the pitch sequence of the original Bach. If the x-values are not rounded at all, then each x_i is unique (for all practical purposes), and no x_i will be associated with more than one pitch. However, when the x_is are rounded to just a few decimal places, it is possible for an x_i to repeat. For instance, the x-value -9.59 of the reference trajectory (for Variation 1) occurs twice such that $x_{58}=x_{106}=-9.59$. Accordingly, x_{58} is associated with $p_{58}=E3$, but x_{106} is paired with $p_{106}=D3$. In the case of two or more pitch choices for a given x-value, the p_i with the largest i is chosen, e.g., the x-value -9.59 is paired with $p_{106}=D3$. This ensures a greater likelihood that a different note from the original will occur in the variation, i.e., $x'_{58} = -9.60 \mapsto D3$ rather than the $E3$ of the original.

It is possible to influence the style of one piece, or part thereof, with that of another. For example, suppose Piece B is appended to Piece A. Their combined pitch sequence becomes the input for the variation technique. The output consists of a variation of AB, called A′B′. However, Piece B′ can be cut away from A′ and used independently. This was done in *Variations to a Theme* where the Theme of Fig. 8 assumed elements not contained in the original Bach Prelude. The variation technique took the Cadenza (A), Chorale (B) and Prelude (C) as input and produced a variation A′B′C′. A′ and B′ were eliminated, leaving the varied Prelude C′ influenced by the musical landscapes of the previous Cadenza and Chorale. The variation C′ inspired the composed Theme given by Fig. 8.

The technique introduced here, applied to the context-dependent works discussed in Sec. III, produces variations that can be analyzed and used musically, suggesting that the method can be generalized to other sequences of context-dependent symbols $\{s_i\}$, $i=1,\ldots,N$. Accordingly, the x axis becomes a symbol axis encompassing a finite number of regions, demarcated by symbols, rather than an infinite number of points. Each region returns one of N possible s_i. The chaotic mapping applied for each x'_j is given by

$$f(x'_j)=s_{g(j)}, \tag{5}$$

where $g(j)$ is defined as above.

The chaotic mapping has also been applied to chaotic trajectories from another chaotic system proposed by Lorenz,[33]

$$\dot{x}=-y^2-z^2-ax+aF, \tag{6}$$

$$\dot{y}=xy-bxz-y+G, \tag{7}$$

$$\dot{z}=bxy+xz-z, \tag{8}$$

where $a=0.25$, $b=4$, $F=8$, and $G=1$, as well as the Rössler[34] system,

$$\dot{x}=-y-z, \tag{9}$$

$$\dot{y}=x+ay, \tag{10}$$

$$\dot{z}=b+z(x-c), \tag{11}$$

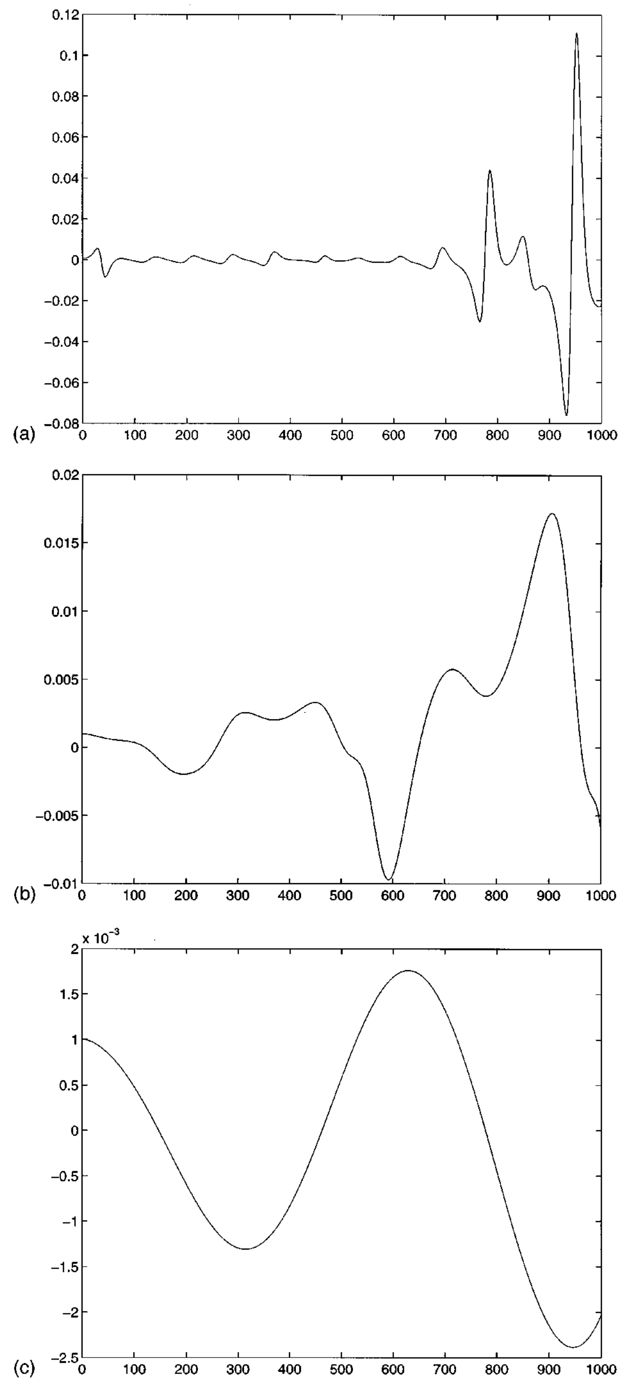

FIG. 9. The difference in x-values (vertical axis) between chaotic trajectories with reference IC (1,1,1) and new IC (0.999, 1, 1) vs. 1000 time steps of the integration (horizontal axis) for (a) Eqs. (1)–(3), (b) Eqs. (6)–(8), and (c) Eqs. (9)–(11). For all trajectories, the integrations are sampled every step with $h=0.01$. All computations are double precision with x-values rounded to six decimal places.

with chaotic parameters $a=b=0.2$ and $c=5$. For the same methods outlined in the caption to Fig. 2, variations resulted which sustained interest.

For each of the three chaotic systems described by Eqs. (1)–(3), Eqs. (6)–(8), and Eqs. (9)–(11), a graph is given in Fig. 9 which plots the difference in x-values between the reference and new trajectories. These graphs suggest that the

Lorenz equations (1)–(3) may be better for variations than the other two systems, at least for close ICs, $h = 0.01$ and an integration length of about 1000 time steps, conditions used to produce the first two Bach variations and the Gershwin variation. Plotting the difference in x-values between reference and new trajectories, as simulated by Eqs. (1)–(3), displays a good balance between positive, negative, and zero excursions. In particular, there are many alternations between the cases where $x_i - x_i' > 0$ and those where $x_i - x_i' < 0$. The repeated presence of positive (negative) excursions ensures that new trajectory x-values are at least falling to the left (right) of the reference x-values and therefore have a chance to trigger original (different) notes. The balance between positive and negative excursions, in conjunction with predominantly small differences in x-values, imply that the tracking mechanism of the chaotic mapping will preserve parts of the original pitch sequence in the variation. However, in the Rössler equations and the Lorenz equations (6)–(8), there are far fewer alternations between these two cases, and, in the latter case, the bounds for the differences in x-values are erratic. Though no definitive conclusions can be drawn on the basis of Fig. 9, plotting the difference in x-values between new and reference trajectories suggests a way of evaluating a chaotic system for its variation potential. The chaotic mapping in tandem with different chaotic systems will be examined more closely in later work.

V. CONCLUSION

A technique has been designed that takes a highly context-dependent application (music) and generates variations, via a chaotic mapping, that retain stylistic ties to the original or mutate beyond recognition, by appropriate choice of the IC. The technique works because nearby orbits can track each other in one or more variables via the chaotic mapping, thereby ensuring portions of the original recur in the variation. If the orbits do not track, pitches result that are still found in the original piece. Thus, links between variation and source exist, regardless of whether the mapping preserves or alters the original sequence. This technique does not generate music or any other kind of data as random events; rather, it creates a rich set of variations on the original that can be further developed and interpreted. Though the method will not flatter fools, it can lead explorers into landscapes where, amidst the familiar, variation and mutation allow wild things to grow.

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APPENDIX A: FURTHER ANALYSIS

The appoggiatura of $m.$ 1 in Variation 1 resolves upward by whole step. This is not how Bach would typically have treated an appoggiatura. In his music, most appoggiaturas resolve downward by step (whole or half) or upward by half step. Still, there are instances where he employs an upward resolution by whole step, e.g., the $g\sharp$ minor Prelude, WTC II, $mm.$ 2, 4, 17, 31, and 42—but they are rare. Such an unusual departure from customary practice creates a need for further confirmation, which is absent in Variation 1. In other words, a good composer writing in the Baroque style would not merely state, and then abandon, an appoggiatura resolving upwards by whole step. Rather, the unusual treatment of the appoggiatura and its resolution would be emphasized elsewhere in the piece, as in fact happens in the $g\sharp$ minor Prelude. Yet even there, though Bach resolves the appoggiatura upward by whole step in $m.$ 2, the resolution occurs on the seventh tone of the V_5^6 chord, which itself tends toward descent—and does so very shortly.

With respect to harmonic progression, Variation 1 follows the original Prelude quite closely. For example, $m.$ 2 and $m.$ 4 can be analyzed as II_2^4 and I, respectively, just as in the original. However, the harmony in $m.$ 2 is colored by the $E4$, an unaccented lower neighbor note* with a delayed resolution to the $F4$ of beat 2. Similarly, the passing tone* on $D3$ in $m.$ 4 (passing from $E3$ in beat 1 to the $C3$ of $m.$ 5) shades $m.$ 4 differently from the original.

The turn in $m.$ 3 of Variation 2 is introduced after the initial sounding of the $F4$ in the third beat, and is considered an unaccented inverted turn involving the notes $E4$, $F4$, $G4$, $F4$. However, this turn is postponed and interrupted. It is postponed by $G3$ which is part of the dominant seventh chord. The $G3$ is prolonged until the interruption by $A3$, which acts as a neighbor note to $G3$, returning to it (in the same voice) after the first beat of $m.$ 4. Because the postponement and interruption occur in a lower voice, the ear is able to hear the effect of the turn in the upper voice.

The harmonic progression of Variation 2 retains the basic harmony of the Bach Prelude, while diverging from it in ways that make the Variation sound as if written much later than the original score. The first half of $m.$ 1 can be interpreted in the tonic by analyzing the $B2$ on the downbeat as an accented lower neighbor (or appoggiatura) to the $C3$ on beat 3, the $F\sharp 3$ as a lower neighbor to $G3$, and the $D3$s as lower neighbors to $E3$. The first 6 sixteenths of $m.$ 1 could also be interpreted as V_6 with $F\sharp 3$ tonicizing G and the $E3$ an appoggiatura (or accented upper neighbor) to $D3$. The harmonic progression of the original score is further altered in $mm.$ 4 and 5 (where the Variation introduces the VI chord[35] a half-measure early, prolonging this harmony in $m.$ 5), $m.$ 7 (where, in beat 4, the addition of the seventh tone creates V_5^6, a departure from the V_6 of the original), $m.$ 11 (where the dominant chord is heard, not on the downbeat, but on the third through seventh sixteenths, followed by VII_7/V—the $G3$ of the third beat functions as an accented neighbor note to $F\sharp$—with a return to the dominant on the fourth beat).

The high A in m. 11 can be heard as an appoggiatura to an implied G, especially if the G is supplied in m. 12.

APPENDIX B: GLOSSARY OF MUSICAL TERMS

An **appoggiatura** is an accented melodic dissonance which resolves by half or whole step, up or down, on a weaker beat or beat division.[36] The appoggiatura often leans towards a specific note of resolution and thus creates an expectation which is fulfilled when it resolves.[37]

A **harmonic progression** is a succession of root chords, represented by roman numerals indicating the scale degrees upon which the chords are built. These chords follow one another in a controlled and orderly way, according to principles of good voice leading.[38] The tonic chord of a piece is designated by I, its supertonic by II, the dominant by V, and the leading tone by VII. The tonic chord is built on the first degree of the scale, the supertonic on the second degree, the dominant on the fifth, and the leading tone on the seventh. Arabic numerals designate the inversion of the chord.

Harmonic rhythm is the rhythmic pattern provided by the changes of root harmony as they occur in a musical composition.[39]

The **inversion** of a series or succession of notes is found by changing each ascending interval into the corresponding descending interval, and vice versa.[39]

The **neighbor note** or auxiliary is an unaccented tone one step above or below a harmonic tone, which returns immediately to the same tone.[40] The neighbor note is not necessarily a dissonant tone. An incomplete neighbor note results when the harmonic tone either does not precede or follow the neighbor note.[38] If the incomplete neighbor note is a step above (step below) the harmonic tone, it will be called an *upper neighbor* (*lower neighbor*) to that note.

Passing tones fill in a melodic skip on all intervening steps, either diatonic or chromatic.[38]

To **retrograde** an ordering is to list it backwards, i.e., the ordering now begins with the last event and ends with the first.[41]

The **retrograde inversion** is the reversal of a series or succession of notes that has been inverted.

A **sequence** is the systematic transposition of a harmonic pattern and its associated melodic and rhythmic patterns to other degrees of the scale. Most theorists agree that a single transposition of such a pattern does not constitute a full sequence, the systematic transposition not having been established until the third occurrence of the initial pattern. Three separate appearances, involving two transpositions, are needed to show that the transposition interval is consistent. A **half-sequence** is one in which only a single transposition of a harmonic pattern, and its associated melodic and rhythmic patterns, occurs. A **harmonic sequence** occurs when a harmonic progression is immediately restated, starting on another degree of the scale.[38]

A **turn** alternates upper and lower neighbor notes with the main note: upper neighbor, main note, lower neighbor, main note. An inverted turn follows the form: lower neighbor, main note, upper neighbor, main note.[36]

A **vamp** is a transitional or accompanimental chord progression of any length, used as filler until a soloist is ready to start or continue.[42]

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[30] For interpreting the musical scores that demonstrate the results, nonmusicians might imagine the horizontal lines and intervening spaces on the scores as a kind of graph paper. The notes could be considered points on the graph. Then, pattern matching could alert the nonmusician to changes between the variation and the original piece.

[31] The $D4$ in the soprano voice of m. 1 is prolonged, thus creating tension, until its relaxation or resolution on $E4$. Though the prolongation is not literally written out, the $D4$—clearly distinct from the lower voices ($E3$, $G3$, $E3$)—is heard as an accented unresolved dissonance until beat 4, when it resolves upwards by whole step.

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