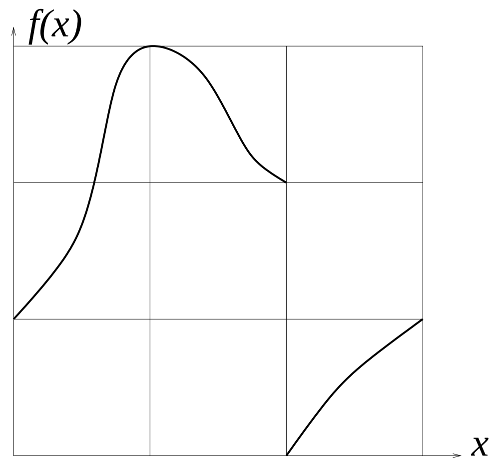
Math 53: Chaos!: Midterm 2, FALL 2009

2 hours, 60 points total, 5 questions worth various points (proportional to blank space)

1. [10 points]

   Consider the function f with the following graph:
   (You may assume f is monotonic in each region)



   (a) Draw the transition graph (use three intervals):

   (b) Which of the following periods can you prove must exist? (give a proof for just *one* of these cases):

   1 2 3 5 6

(c) Prove that a period-4 orbit *cannot* exist. [Hint: consider monotonicity of f^2 in some subinterval]

[BONUS: what periods above 6 must exist and why?]

2. [9 points] Consider, on the unit square, the linear torus map $T(\mathbf{x}) = A\mathbf{x} \pmod 1$, where A is a 2×2 matrix with integer entries.

(a) In the case $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, compute all Lyapunov exponent(s) of the map.

(b) Now for the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, is $\mathbf{0}$ a source/sink/saddle? [Hint: consider the action of A^n on the point $(0, \varepsilon)$ for arbitrarily small ε]. Compute the Lyapunov exponent(s) and explain the discrepancy between this and whether $\mathbf{0}$ is a sensitive point.

(c) Prove that, for general A, if the map T is area-preserving (hence invertible) then the Lyapunov exponents of T and T^{-1} are the same.

3. [14 points] Consider 1D motion of a point particle in the potential $P(x) = x^2 - x^4$.

(a) Write a system of first-order ODEs for the dynamics in this potential, with no damping.

(b) Graph the potential function (careful about signs) and below that, graph the phase plane $(x, \dot{x})$ showing several orbits which show all the types of motion that can occur:

(c) Find all equilibria and categorize their stability. Justify your stabilities by giving rigorous arguments. [Hint: use the phase plane]

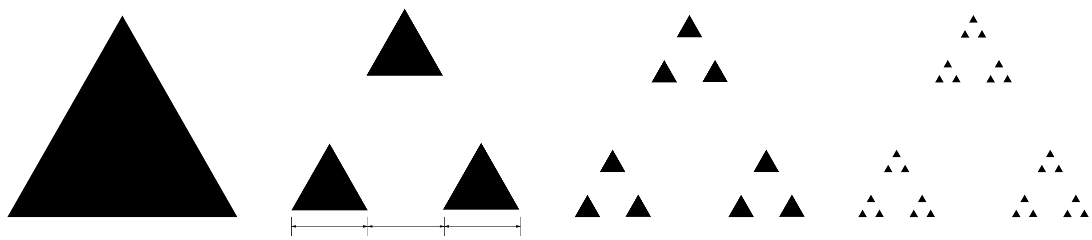
(d) What is the allowable energy range where bounded motion can happen? (give upper and lower limits)

(e) Imagine a small amount of damping is now added. Sketch on a phase plane the basin of the stable equilibrium.

[BONUS] Give a bound on the speed of a particle which passes through the stable equilibrium more than once.

4. [13 points]

(a) Find the box-counting dimension of the 'triangular Sierpinski carpet' set given by the limit of the process shown applied to the equilateral triangle: (in each step the three lengths as shown are equal)



(b) Describe a probabilistic game whose attractor is the above fractal. (You may use words rather than equations, but be clear and concise.)

(c) Find the box-counting dimension of the set of initial values whose orbits remain bounded for all time, under the one-dimensional map

$$f(x) = \begin{cases} 4x, & x \leq 1/2 \\ 4(1-x), & x > 1/2 \end{cases}$$

[Hint: graph f. Partial credit given for describing the type of set.]

(d) Does the set in (c) contain a finite, countably infinite, or an uncountably infinite number of points? [BONUS: prove your answer]

(e) Give an example of a sequence of box sizes ε tending to zero that would *not* be appropriate for computing box-counting dimension.

5. [14 points] Random short-answer questions

   (a) Among a set of 10^4 points there are 10^5 pairs of points lying within Euclidean distance 0.1 of each other, but only 10^2 pairs lying within distance 0.001 of each other. Use this to estimate the correlation dimension of the set.

(b) Consider the maps $f(x) = 4x(1-x)$ and $g(x) = 2 - x^2$. Prove that they are conjugate under the (linear) bijection $y = C(x) = 4x - 2$. If the Lyapunov exponent of f is $\ln 2$, what can you deduce (if anything) about the Lyapunov exponent of g?

(c) Characterize the set of all left-endpoints remaining in the middle-thirds Cantor set using the ternary system. Is this set countably or uncountably infinite?

(d) Is the middle-thirds Cantor set dense in [0,1]?

(e) Is the point -2 in the Mandelbrot set?

(f) Is the point i in the Julia set for $c = -1$?