Math 53: Chaos! 2009: Midterm 1

2 hours, 54 points total, 6 questions worth various points (proportional to blank space)

1. [9 points] Consider the two-dimensional map $\mathbf{x} \to A\mathbf{x}$.

   (a) If $A = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 0 \end{bmatrix}$, describe the object formed by applying the map to the unit disc $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1\}$. Include all relevant lengths and directions (unnormalized direction vectors are fine).

   (b) For this A, do any points in the unit disc get mapped outside the unit disc?

   (c) For this A, find the fixed point(s) of the map and classify them.

(d) Now if $A = \begin{bmatrix} 9/2 & -4 \\ 2 & -3/2 \end{bmatrix}$, does the map have any points with *sensitive dependence*? If so, give a proof for one such point. If not, explain why and categorize any fixed point(s). [Partial credit given for correct definition of sensitive dependence].

2. [10 points] Consider the two-dimensional map $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ a - y^2 \end{pmatrix}$

(a) Solve for all fixed points of f. For what range of a do (real) fixed points exist?

(b) Fix $a = 0$, and for each of the two fixed points, answer: is it hyperbolic? Can you deduce if it is a sink, source, or saddle? [Hint: first find the y values].

FIXED POINT 1:

FIXED POINT 2:

(c) Find the critical value of a above which both fixed points are of the same type.

3. [10 points] Consider the $f(x) = 3x \pmod 1$ which maps the interval $[0, 1)$ to itself.

(a) $x_0 = \frac{3}{26}$ is a fixed point of period k. Find k

(b) Is this a periodic sink, periodic source, or neither? (show your calculation)

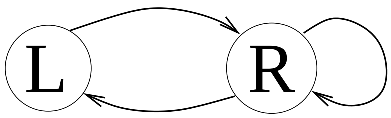
(c) How many fixed points of f^2 are there in $[0, 1)$?

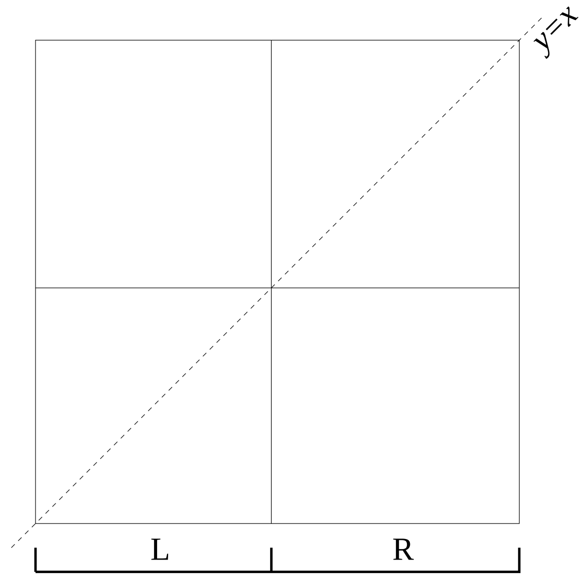
(d) Prove that if an orbit $\{x_0, x_1, \ldots\}$ is eventually periodic, then x_0 is rational.

(e) Compute the Lyapunov *exponent* (not number) of such an eventually periodic orbit, and use this to estimate how many iterations will it take for an initial computer rounding error of 10^{-16} to reach size 1?

4. [11 points]

(a) Draw a possible graph of a smooth continuous function f mapping $L \cup R$ to itself, with only one turning point, whose transition graph is that shown below. Use the axes and intervals shown to the right. (Be sure to check your f has the correct transition graph).





(b) Prove that f has no fixed points in L.

(c) Prove that there exist orbits which are not fixed, periodic, or eventually periodic.

(d) Show the subdivision down to level 4 (that is, the correct ordering of all 4-symbol itinerary subintervals on $L \cup R$). Take plenty of horizontal space. How many subintervals are there? [Hint: you can answer the latter without the former]

(e) BONUS: Derive a formula for the number of subintervals at level k.

5. [6 points] Consider $T(\mathbf{x}) = A\mathbf{x} \pmod 1$, where $A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, acting on the torus $\mathbf{x} \in \mathbb{T}^2 = [0,1)^2$.

(a) Does the map T have an inverse? (explain using properties of the map)

(b) Find all fixed points of T in the torus.

(c) Answer (b) for the case of $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

6. [8 points] Random short questions.

(a) The origin is a fixed point of $f(x) = \tan x$. Categorize it as a source, sink, or neither. [BONUS: Prove your answer].

(b) A map $f : \mathbb{R} \to \mathbb{R}$ has $f^6(x) = x$. What are the possible periods of x as a periodic fixed point, if any?

(c) Give a precise mathematical definition of the *basin* of a fixed point p.

(d) Explain in a sentence what a period-doubling bifurcation is (include a sketch of a bifurcation diagram with axes).