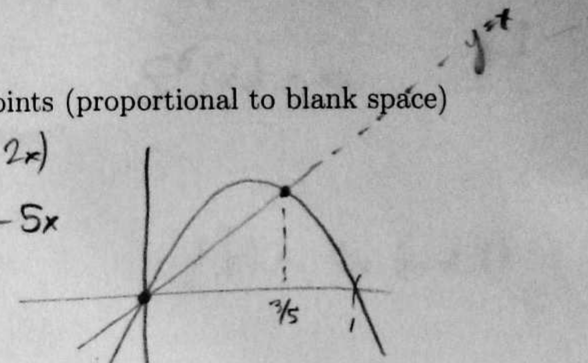
Barnett
10/23/07

SOLUTIONS

Math 53: Chaos!: Midterm 1

2 hours, 60 points total, 6 questions worth various points (proportional to blank space)

$f' = \frac{5}{2}(1-2x) = \frac{5}{2} - 5x$



[3] 1. [8 points] Consider the map $f(x) = \frac{5}{2}x(1-x)$.

(a) Find the fixed points and their stability.

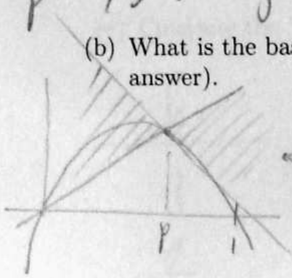
$f(p) = p$

$\frac{5}{2}p - \frac{5}{2}p^2 = p \Rightarrow -5p^2 = -3p \Rightarrow p = 0, 3/5$

$p=0$: $f'(0) = \frac{5}{2} > 0 \Rightarrow$ source

$p = 3/5$: $f'(3/5) = \frac{5}{2} - 5\cdot\frac{3}{5} = -1/2 \quad |f'|<1 \Rightarrow$ sink

[2] (b) What is the basin of the nonzero fixed point? (Try to find the maximal such set, and prove your answer).



all points in $(3/5, 1)$ map to values in $(0, 3/5)$

'butterfly' region where graph implies basin.

all points $x \in (0, 3/5)$ have $|f(x - 3/5)| < |x - 3/5|$

ie move closer to $p = 3/5$ each iteration.

$\Rightarrow$ basin is $(0,1)$.

Also $x \leq 0$ either heads to $-\infty$ or stays at 0.
$x = 1$ heads to 0.

[1] (c) Find an *eventually periodic* point which however is not periodic or fixed.

$x_0 = 2/5 \xrightarrow{f} 3/5 \xrightarrow{f} 3/5 \rightarrow \cdots$ (period-1)

$x_0 = 1 \xrightarrow{f} 0 \xrightarrow{f} 0 \rightarrow \cdots$

[2] (d) Find the Lyapunov *exponent* (not number) of all orbits that do not tend to infinity (or zero).

From (b) all such points are asymptotically periodic to $p = 3/5$ fixed pt.

$\Rightarrow$ They share its Lyapunov exponent, which is $h(3/5) = \ln|f'(3/5)|$

$= \ln(1/2) = -\ln 2$.

[Note: strictly any point ever hitting $\frac{1}{2}$ should be excluded]

2. [14 points] Consider the map $f(x) = 2x$ (mod 1) on $x \in [0, 1)$.

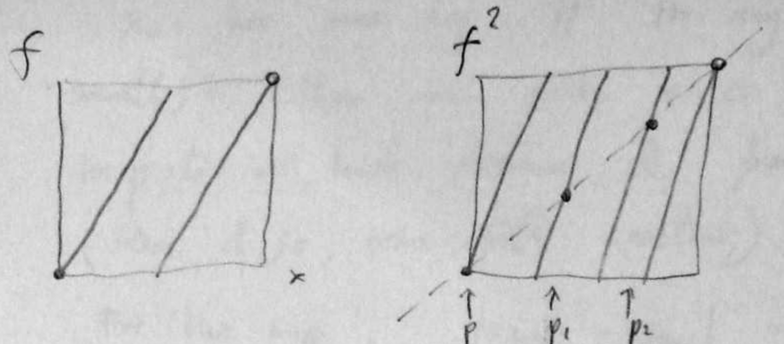
period is minimum k such that $f^k(x_0) = x_0$, ie 3.

[1] (a) Is $x_0 = 1/7$ a period-6 point? (Explain). If not, what, if any, is the period of this orbit?

$$1/7 \to 2/7 \to 4/7 \to 1/7 \to 2/7 \to 4/7 \to 1/7$$

$$f^6(x_0) = x_0$$

[2] (b) Sketch a graph of $f^2(x)$. How many fixed points are there?



$f^2(x) = 4x$ (mod 1)

3 fixed points (since $4-1$)

[4] (c) Compute the 'periodic table' (*i.e.* how many period-k orbits there are for each k) up to $k = 5$.

k	# fixed pts of f^k	# accounted for by lower periods	# PO's
1	1	0	1
2	3	1	1
3	7	1	2
4	15	3	3
5	31	1	6
⋮	⋮ $2^k - 1$		

[3] (d) Using any method you prefer, prove that the map has periodic orbits of *all* periods.

i) Itineraries: L on $[0, 1/2)$, R on $[1/2, 1)$ transition graph is complete

so any periodic sequence eg. $\overline{LRLRR}$ is possible, gives periodic orbit.

ii) The 3rd column of periodic table has upper bound given by sum of #'s lower fixed points: ie $(2^{k-1}-1) + (2^{k-2}-1) + \cdots 1$

$= 2^k - 2 - (k-1)$ which is never as large as $2^k - 1$ ⇒ must be PO of k.

[3] (e) State the mathematical definition of a point having *sensitive dependence.* Prove that all points in $[0,1)$ have this property.

x_0 has sens. dep. if for any $\varepsilon > 0$, no matter how small, there are points $x \in N_\varepsilon(x_0)$ which eventually map to at least distance d from wherever x_0 goes. (Here d is some $O(1)$ constant).

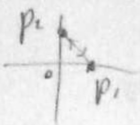
For this map, $|x_{k+1} - y_{k+1}| = 2|x_k - y_k|$ if the domain is connected into a loop $\bigcirc^{0}_{1}$, i.e. distance doubles.

So for any ε, $2^k \varepsilon > d$ for some sufficiently large k.

[You may also use the 4 subitineraries LL LR RL RR.

(f) BONUS: what happens if the computer is used to numerically iterate starting at $x_0 = 1/7$?

$1/7$ is not stored exactly, so after 50 or so iterations the orbit leaves the unstable fixed pt, goes chaotic.



3. [8 points]

period-2, I meant

(a) The point $(3/5, 0)$ is a period-two fixed point for the Hénon map $\mathbf{f}(x,y) = (a - x^2 + by, x)$ with parameters $a = 9/25$, $b = 2/5$. Is this point a sink, source, saddle? Is it hyperbolic? in the period-2 sense again!

[4] $\vec{p}_1 = \begin{pmatrix} 3/5 \\ 0 \end{pmatrix}$ $\quad \vec{p}_2 = \vec{f}(\vec{p}_1) = \begin{pmatrix} 9/25 - (3/5)^2 + 2/5 \cdot 0 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix}$

$Df^2(p_1) = Df(p_2) \cdot Df(p_1)$ $\quad$ with $Df = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$

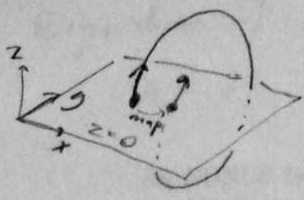
Jacobean $\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$

$= \begin{pmatrix} 0 & 2/5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -6/5 & 2/5 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2/5 & 0 \\ -6/5 & 2/5 \end{pmatrix} \longrightarrow$ eigenvalues $\lambda = +2/5$ twice.

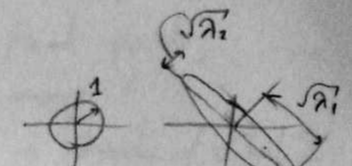
$\rightarrow$ a <u>sink</u>, & <u>hyperbolic</u>.

[4] (b) Explain in 1-2 sentences the concept of a Poincaré map.



Say an ODE (continuous in time) has trajectory
$\vec{x}(t) = (x, y, z) \in \mathbb{R}^3$

A Poincaré map is the map between successive crossings of same surface in $\mathbb{R}^3$, passing in the same orientation (eg $\dot{z} > 0$ only, hitting the $z=0$ plane). It is therefore a discrete in time map of one lower dimension than the original space.



4. [9 points] Consider the linear map on $\mathbb{R}^2$ defined by the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

[5] (a) Describe the object formed by applying the map to the unit disc $\{\mathbf{x} : |\mathbf{x}| < 1\}$. Include all relevant lengths and directions (unnormalized direction vectors are fine). ie don't waste time normalizing them.

It is an ellipse.

$$AA^T = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Diagonalize: $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 - 4 = 0$ $\quad \lambda = +3 \pm \sqrt{9-1} = 3 \pm \sqrt{8}$

Eigenvector for $\lambda_1 = 3 + \sqrt{8}$: $\begin{bmatrix} 5-3-\sqrt{8} & 2 \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \end{bmatrix}$ so $(2-\sqrt{8})v_1 = 2v_2$
$v_2 = (1-\sqrt{2})v_1$

semimajor axis $= \sqrt{3+\sqrt{8}} = 1 + \sqrt{2}$, direction $\longrightarrow \vec{v} = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$

" minor axis $= \sqrt{3-\sqrt{8}} = \sqrt{2} - 1$, " $\begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$ since must be $\perp \vec{v}$ (matrix AA^T is symm.)

[1] (b) What is the area of this object?

Area $= |\det A| \cdot$ (area of unit disc)
$= (+1) \cdot \pi = \pi$.

eg. not a source since there are points arb. close to $\vec{0}$ which do not ever leave the neighborhood.

[3] (c) Describe the stability of the fixed point $(0,0)$ (sink, source, saddle, hyperbolic?)

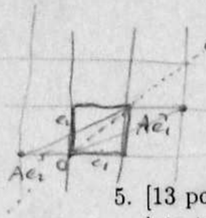
Eigenvalues of $\underline{A}$ itself control this: $\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$

$\lambda = +1$ twice so not hyperbolic, also not a source, sink or saddle.

(d) BONUS: how many fixed points does the torus map $A\mathbf{x}$ (mod 1) have, and what form are they? (don't remember, rather, work it through. What property in general leads to this weirdness?).

fixed pts given by
$$2x - y = x + n$$
$$x = y + m$$
accounts for the mod 1, $n, m \in \mathbb{Z}$

ie
$$x - y = n$$
$$x - y = m$$
so $n = m$, and only solutions in $\mathbb{T}^2$ are $n = m = 0$.

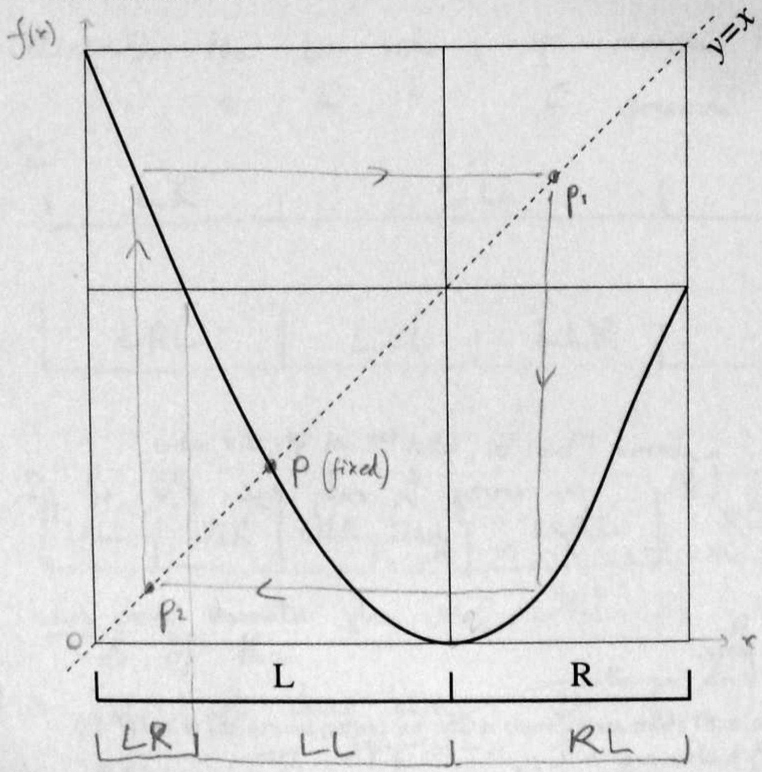


fixed pts lie on line.

All points with $x = y$ are fixed points, so there are infinitely many!

the problem was that 1 was eigenvalue of $A \Rightarrow$ formulae of Chl. 2 fail.

5. [13 points] Consider the 1D map given by the following graph of $f(x)$ on $[0,1]$, which has been split into two intervals.



showing the only period-2 orbit $\overline{LR}$

NB: <u>Proof</u> that $\overline{LR}$ can corresp. to only <u>one</u> PO of p2:

using: f monot. in L, & in R.

$\Rightarrow f^2$ monot. (decr.) in LR. $\Rightarrow$ only intersects $y=x$ <u>once</u>.

$\Rightarrow$ any period $2k$ would need $\{p_1, \cdots p_k\} \in LR$, $p_1 < p_2 \cdots < p_k$

but $f^2\{p_1 \cdots p_k\}$ = some permutation of $\{p_1 \cdots p_k\}$

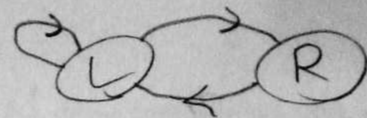
contradiction since f^2 monot. means $p_i < p_j \rightarrow f^2(p_i) > f^2(p_j)$

$\Rightarrow k=1$ (ie p2) is only possible.

[2] (a) Give the itinerary for the only period-two orbit. $\overline{LR}$ eg draw it.

Since : i) You cannot stay in R for more than one it at a time.
ii) There is only 1 orbit staying in I : the fixed pt. p

[2] (b) Draw the transition graph for f.



reading from graph : $\begin{cases} f(L) = [0,1] \\ f(R) = L \end{cases}$

[2] (c) Sketch roughly where the subinterval LR is and show to which subinterval it is mapped under f.

see graph $\quad f(LR) = R \quad$ since bites off the first symbol. ('symbol shift').

[5] (d) Show the subdivision down to level 4 (that is, the correct ordering of all 4-symbol itinerary subintervals on $[0, 1]$). How many subintervals are there? R must always cause L (no split).

On the L side, f reverses the order of intervals
" R " f preserves the " " "

level 2: | LR | LL | RL | eg get by looking

level 3: | LRL | LLL | LLR | RLR | RLL |

when bite off the 1st letter, it's level 3 reversed — when bite off 1st letter, it's L side of level 3 in same order.

level 4 | LRLL | LRLR | LLLR | LLLL | LLRL | RLRL | RLLL | RLLR |

8 of them.

[2] (e) What is the *lowest* period for which there exists more than one periodic orbit? (show why) — note we don't count cyclic permutation as distinct: $\overline{LR}$ is same p2 orbit as $\overline{RL}$ [since f^2 monot. decr. on LR, only 1 p2 possible] — is guaranteed to (strictly)

Tricky one

p2 : $\overline{LR}$

p3 : $\overline{LLR}$ since $\overline{LRR}$ impossible

p4 : $\overline{LLLR}$ since $\overline{LLRR}$ & $\overline{LRRR}$ impossible, $\overline{LRLR} = \overline{LR}$ which ~~may~~ be p2.

→ p5 : $\overline{LRLLR}$ & $\overline{LLLLR}$ are <u>different</u> p5 orbits (guaranteed)

F_1 F_2 F_3 F_4 F_5 F_6
1 1 2 3 5 8 13 21 ...

(f) BONUS: how many subintervals are there at level k?

F_{k+2} where F_k is kth Fibonacci number.

Prove using transition graph: $F_k = F_{k-1} + F_{k-2}$

6. [8 points]

[4] (a) Find $\lim_{n\to\infty} A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. If the limit does not exist, give a vector *direction* that the sequence of vectors approaches.

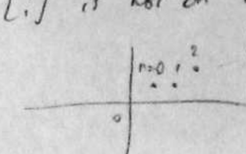
golden ratio
$\phi = 1.618\ldots$, $0.618\ldots$

eigvals: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

It's a saddle so no limit exists unless $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ on stable manifold.

eigvec. of larger eigenvalue: $\begin{pmatrix} 1-\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \end{pmatrix}$ so $\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)v_1 + v_2 = 0$

$\vec{v}$ = direction vector $= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$ ← so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not on unstable manifold



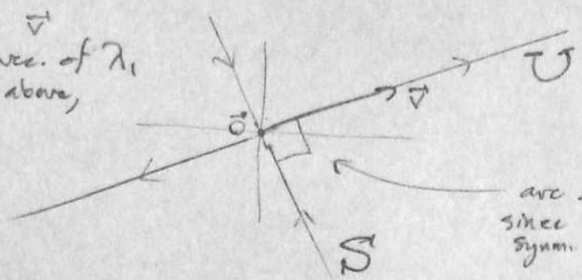
heads to ∞ in this direction, growing by factor ϕ asymptotically.

[2] (b) What type of fixed point is **0** under the map given by A?

Saddle, hyperbolic, since $|\lambda_1| > 1$ & $|\lambda_2| < 1$.

[2] (c) Find the stable and unstable manifolds for the fixed point **0**. ... which is possible since a saddle

using $\vec{v}$ eigenvec. of λ_1 from above,



are $\perp$ since A symm.

$U = \{\alpha\vec{v} : \alpha \in \mathbb{R}\}$

S = orthog. complement of U (since A symmetric)

$= \{\vec{u} : \vec{u}^T\vec{v} = 0\}$

Alternatively S is the span of the eigenvector with $\lambda_2 < 0$.

(d) BONUS: explain the Fibonacci connection.

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ ← Fibonacci numbers as above.