Math 53: Chaos!: Midterm 1

2 hours, 60 points total, 6 questions worth various points (proportional to blank space)

1. [8 points] Consider the map $f(x) = \frac{5}{2}x(1-x)$.

   (a) Find the fixed points and their stability.

   (b) What is the basin of the nonzero fixed point? (Try to find the maximal such set, and prove your answer).

   (c) Find an *eventually periodic* point which however is not periodic or fixed.

   (d) Find the Lyapunov *exponent* (not number) of all orbits that do not tend to infinity (or zero).

2. [14 points] Consider the map $f(x) = 2x \pmod 1$ on $x \in [0, 1)$.

   (a) Is $x_0 = 1/7$ a period-6 point? (Explain). If not, what, if any, is the period of this orbit?

   (b) Sketch a graph of $f^2(x)$. How many fixed points are there?

   (c) Compute the 'periodic table' (*i.e.* how many period-k orbits there are for each k) up to $k = 5$.

   (d) Using any method you prefer, prove that the map has periodic orbits of *all* periods.

(e) State the mathematical definition of a point having *sensitive dependence*. Prove that all points in $[0, 1)$ have this property.

(f) BONUS: what happens if the computer is used to numerically iterate starting at $x_0 = 1/7$?

3. [8 points]

(a) The point $(3/5, 0)$ is a period-two fixed point for the Hénon map $\mathbf{f}(x, y) = (a - x^2 + by, x)$ with parameters $a = 9/25$, $b = 2/5$. Is this point a sink, source, saddle? Is it hyperbolic?

(b) Explain in 1-2 sentences the concept of a Poincaré map.

4. [9 points] Consider the linear map on $\mathbb{R}^2$ defined by the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

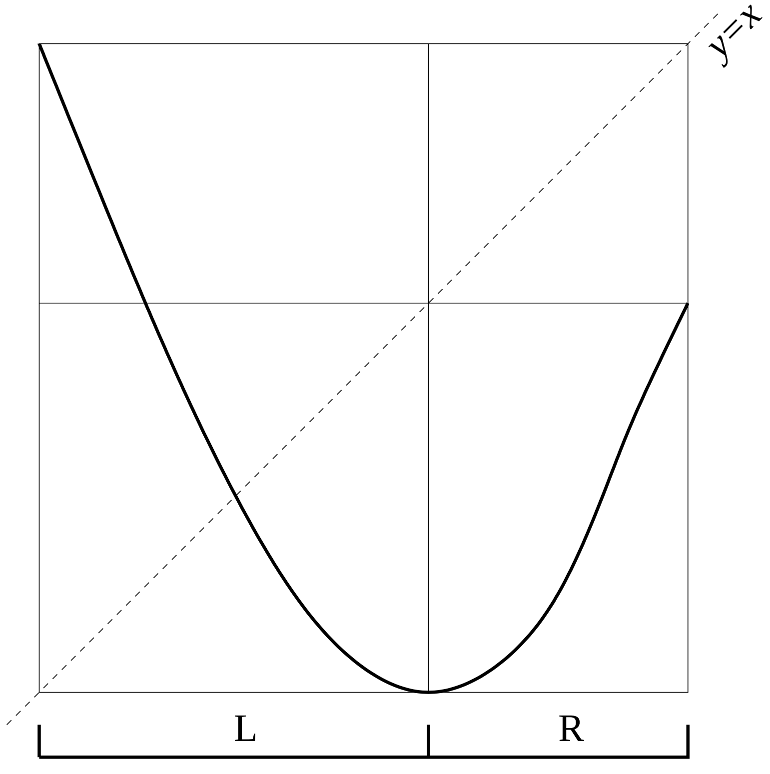
(a) Describe the object formed by applying the map to the unit disc $\{\mathbf{x} : |\mathbf{x}| < 1\}$. Include all relevant lengths and directions (unnormalized direction vectors are fine).

(b) What is the area of this object?

(c) Describe the stability of the fixed point $(0, 0)$ (sink, source, saddle, hyperbolic?)

(d) BONUS: how many fixed points does the torus map $A\mathbf{x} \pmod 1$ have, and what form are they? (don't remember, rather, work it through. What property in general leads to this weirdness?).

5. [13 points] Consider the 1D map given by the following graph of $f(x)$ on $[0, 1]$, which has been split into two intervals.



(a) Give the itinerary for the only period-two orbit.

(b) Draw the transition graph for f.

(c) Sketch roughly where the subinterval LR is and show to which subinterval it is mapped under f.

(d) Show the subdivision down to level 4 (that is, the correct ordering of all 4-symbol itinerary subintervals on $[0, 1]$). How many subintervals are there?

(e) What is the *lowest* period for which there exists more than one periodic orbit? (show why)

(f) BONUS: how many subintervals are there at level k?

6. [8 points]

(a) Find $\lim_{n\to\infty} A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. If the limit does not exist, give a vector *direction* that the sequence of vectors approaches.

(b) What type of fixed point is $\mathbf{0}$ under the map given by A?

(c) Find the stable and unstable manifolds for the fixed point $\mathbf{0}$.

(d) BONUS: explain the Fibonacci connection.