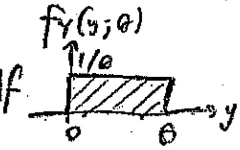
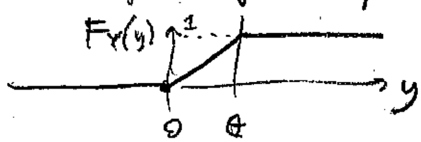
MATH 50 WORKSHEET : Consistency of an estimator — SOLUTIONS

2/8/06
Barnett

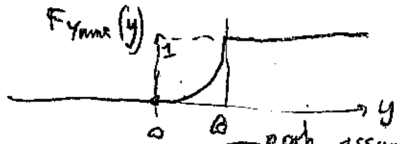
Consider $\hat{\theta}_n = Y_{max}$ the ML estimator for θ of uniform pdf



Find the cdf for single sample $F_Y(y) = y/\theta$ for $0 \le y \le \theta$



Use this to write cdf of estimator $F_{Y_{max}}(y) = \begin{cases} 0 & y \le 0 \\ \left(\frac{y}{\theta}\right)^n & 0 \le y \le \theta \\ 1 & y > \theta \end{cases}$



prob assuming data drawn from true param value θ.

Use this to write $P(|\hat{\theta}_n - \theta| < \varepsilon) = P(\theta - \varepsilon \le \hat{\theta}_n < \theta + \varepsilon)$ but note $\hat{\theta}_n$ cannot exceed θ!

$= P(\theta - \varepsilon < \hat{\theta}_n) = 1 - F_{Y_{max}}(\theta - \varepsilon) = 1 - \left(\frac{\theta - \varepsilon}{\theta}\right)^n$

Prove this has a limit as $n \to \infty$. What is it? limit is $\underline{1}$.

$\frac{\theta-\varepsilon}{\theta} < 1$ so $\left(\frac{\theta-\varepsilon}{\theta}\right)^n \to 0$ as $n \to \infty$.

Is this $\hat{\theta}_n$ <u>consistent</u>? Yes, since limit is $\underline{1}$: all prob. mass of $\hat{\theta}_n$ concentrates at θ asymptotically.

If $\theta = 4$, $\varepsilon = 0.1$, $\delta = 0.2$, how large must n be so that $p(|\hat{\theta}_n - \theta| < \varepsilon) > 1 - \delta$?

(Solve for general ε, δ then subst. at end)

Ie at least 80% prob. of being within 0.1 of truth?

as before, the prob is $1 - \left(\frac{\theta - \varepsilon}{\theta}\right)^n$ which must exceed $1 - \delta$.

$\Rightarrow$ critical n given by $1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^n = 1 - \delta$

take logs $\Rightarrow$ $n \ln \frac{\theta - \varepsilon}{\theta} = \ln \delta$

$\rightarrow$ $n = \frac{\ln \delta}{\ln\left(1 - \frac{\varepsilon}{\theta}\right)} = \frac{\ln 0.2}{\ln\left(1 - \frac{0.1}{4}\right)} = 63.57$

so $\boxed{n \ge 64 \text{ is enough}}$

Is $\hat{\theta}_n = \frac{n+1}{n} Y_{max}$ a consistent estimator? Yes since correction tends to 1 as $n \to \infty$. How about $2\bar{Y}$? yes also, since variance of $2\bar{Y} \to 0$ as $n \to \infty$, unbiased