Expected values - 1/11/06

Binomial distribution (in book)

If random variable X is the number of heads in n tosses of a coin with bias $0 \leq \alpha \leq 1$, it has binomial distribution

$$p_X(k) = \binom{n}{k} \alpha^k (1-\alpha)^{n-k} =: B_n(k)$$

Expected value

$$\begin{aligned}
E(X) &:= \sum_{k=0}^{n} k p_X(k) \\
&= \sum_{k=1}^{n} k p_X(k) \qquad \text{since the } k=0 \text{ term contributes zero to sum} \\
&= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} \alpha^k (1-\alpha)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \alpha^k (1-\alpha)^{n-k} \qquad \text{cancelling with } k! \\
&= n\alpha \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} \alpha^{k-1} (1-\alpha)^{n-k} \qquad \text{bring out factors of } n \text{ and } \alpha \\
&= n\alpha \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \alpha^{j} (1-\alpha)^{n-1-j} \qquad \text{substitute } j = k-1 \text{ or } k = j+1 \\
&= n\alpha \sum_{j=0}^{n-1} B_{n-1}(j) \\
&= n\alpha
\end{aligned}$$

Note how the limits of the sum changed by 1 when we changes k to j. The last step we recognized the sum as the sum of probabilities of the binomial with j heads out of $n-1$ tosses. This is a standard mathematical 'trick' (it works to get the variance too, although this is a bit more messy).

So, a coin of bias $\alpha = 0.7$ tossed $n = 100$ times has an expected number of heads of $0.7 \times 100 = 70$, not surprisingly.

Hypergeometric distribution (not in book... you're gonna like it!)

Random variable X is the number of red chips in a draw of n without replacement from urn of $N = r + w$ red and white chips.

We start with an identity which can be verified by direct differentiation,

$$\left[\frac{d}{d\mu}(1+\mu)^r\right]\cdot(1+\mu)^w \quad = \quad r(1+\mu)^{N-1} \tag{1}$$

Now the coefficient of μ^{k-1} appearing in the first LHS factor is $k\binom{r}{k}$ (here you must binomial expand *before* taking the derivative). The coeff of μ^{n-k} in the second factor is $\binom{w}{n-k}$. The coeff of μ^{n-1} in the LHS (the product of the two factors) is then the following sum over k (just like your proof of LM3 3.3.10). But this must equal the coeff of μ^{n-1} in the RHS, thus

$$\sum_{k=0}^{n} k\binom{r}{k}\binom{w}{n-k} \quad = \quad r\binom{N-1}{n-1}. \tag{2}$$

Dividing both sides by $\binom{N}{n}$ turns the LHS into what we want,

$$\begin{aligned} E(X) \quad &= \quad \sum_{k=0}^{n} k\frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}} \;=\; r\frac{\binom{N-1}{n-1}}{\binom{N}{n}} \\ &= \quad r\frac{n}{N} \;=\; n\frac{r}{N}. \end{aligned} \tag{3}$$

Since r/N is the fraction of red chips in the urn, which is analogous to α for the binomial, the expectation has exactly the same form as $n\alpha$ for the binomial. How cool is that!

For example, the expected number of red chips when 2 are drawn from an urn of $r = 3$, $w = 2$, is $4/5$.

How does this change if the chips are drawn *with replacement*?