Math 50 Stat Inf: Homework 6—selected solutions

due Wed Feb 15

5.4.18 : variance of first is 5^2 times smaller (more efficient) since $\text{Var}(Y_{max}) = \text{Var}(Y_{min})$.

A. $\sqrt{\text{Var}(\hat{p})} = 0.0014$, (*i.e.* std dev of mean k was 100 times this); you should get z-values of order -2 to +2, *i.e.* from standard normal, otherwise something's wrong!

B. Following calculation of bias in the μ unknown case from class, a simpler version gives bias of $n/(n-1)$ for what's given. The cure is to return to the 'naive' estimate $\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)^2$.

5.7.2 : To show consistent you may EITHER,

1. use Chebyshev's law of large numbers since estimator is the mean of indep samples Y_i^2. The theorem only applies if the variance of each variable is *finite*. Thus you need to prove $\text{Var}(X^2) = E(X^4) + (E(X^2))^2 < \infty$. This follows since the second term is σ^4 and the first can be found using the integral $\int_{-\infty}^{\infty} x^4 e^{-x^2/2} dx = 3\sqrt{2\pi}$, OR
2. Show the mean is correct, and that the estimator's variance, which is $\text{Var}(X^2)/n$, goes to zero. This boils down to same estimate as above.

C. The key fact is the Cauchy has infinite variance, so Chebyshev's law of large numbers bearsk down, and the sample mean never converges! (Cool, eh?) However a ML estimate (or, better, full $L(\mu)$ function as you plotted) does. This is good evidence Bayesian methods are the best.

Run this code to make the required plots:

```
% HW6: Alex qu C - Cauchy pdf sample mean estimator expt

% i)
y = tan(pi*rand(1,10000) - pi/2);
dx = 0.1;    % choose spacing for histogram
x = -4:dx:4;
figure; hist(y, x); xlabel y; ylabel freq; axis([-3 3 0 400]);
% compare to true Cauchy... (optional)
hold on; plot(x, (dx/pi)*ns(end)./(1+x.^2), '-');

% ii)
ns = 10.^(1:7);      % list of n values (note 10^7 causes a few sec wait).
clear m                   % so any old values of m forgotten
for i=1:7
  n = ns(i);                        % do each n value in turn
  y = tan(pi*rand(1,n) - pi/2);     % note: fresh samples each time (optional)
  m(i) = mean(y);                   % note: we fill a list, plot only at end
end
figure; semilogx(ns, m, '+-'); xlabel n; ylabel('sample mean');

% iii)             posterior; we can reuse the y list already sitting there
```

```
n = 100;
m = -2:0.01:2;          % choose list of mu values to calc L over (then plot)
L = ones(size(m));
for i=1:n
  L = L .* (1/pi)./(1+(y(i)-m).^2);     % note ./ .^ since m is a list
end
figure; plot(m, L, '-'); xlabel \mu; ylabel L(\mu);
```

D. $L(\alpha|X) = cp^7(1-p)^3$.

   i) beta with params $r = 8, s = 4$ so $p(\alpha|X) = \frac{\Gamma(12)}{\Gamma(8)\Gamma(4)} p^7(1-p)^3$.

   ii) beta with params $r = 8+2-1, s = 4+4-1$ so $p(\alpha|X) = \frac{\Gamma(17)}{\Gamma(9)\Gamma(7)} p^8(1-p)^6$.

5.8.1 : Ignore the Γ factors in the book's bad explanation. The posterior is a beta with params $r+1$ and $s+k$, so the normalization is $\Gamma(r+s+k+1)/\Gamma(r+1)\Gamma(s+k)$.