Open-Glottis Equilibria in a Two-Mass Vocal Fold Model

Emily A. Golitzin

Adviser: Nishant Malik || Math 46

Introduction

The nonlinear dynamics of vocal fold oscillations have been an object of study for the past several decades. Ishizaka & Flanagan [1] developed the well-known two-mass model, in which the vocal folds are modeled as a four-dimensional system comprising a pair of two spring-coupled, damped oscillating masses constrained to move in the lateral direction. Energy is transferred from the airstream to the vocal cords through a phase difference in the oscillations of the upper and lower masses.

In this project, I analyze the symmetric vocal fold model with linear damping and restitution forces. Using simplified hydrodynamics—assuming quasisteady glottal flow, with Bernoulli flow at the narrowest portion of the glottis and zero supraglottal pressure; neglecting effects of coupling and resonance of the vocal tract; and assuming a pressure drop causing a *vena* contracta at the inlet to the glottis—has been justified by Lucero [2] and Steinecke & Herzel [3], among others, who note that these and other simplifications produce analytical results in line with recorded data. This suggests that the two-mass model, although simplified, provides a reasonable starting point for mathematical analysis of vocal fold dynamics.

Physical Motivation of Model

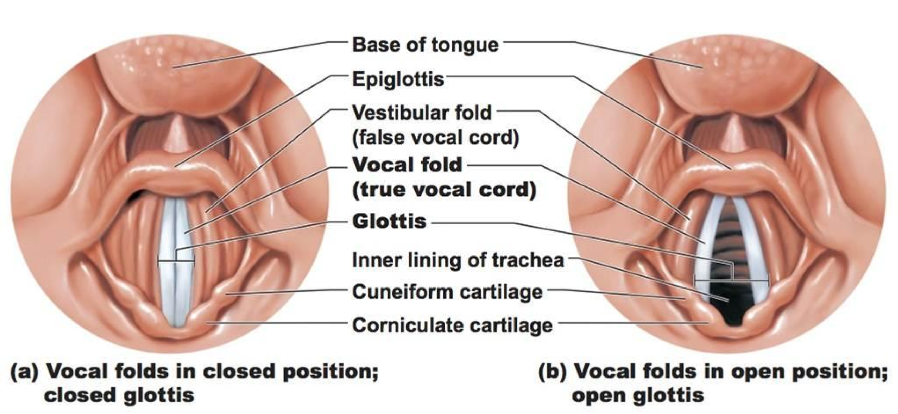


Figure 1: Illustration of the vocal folds, showing adducted and abducted positions [4].

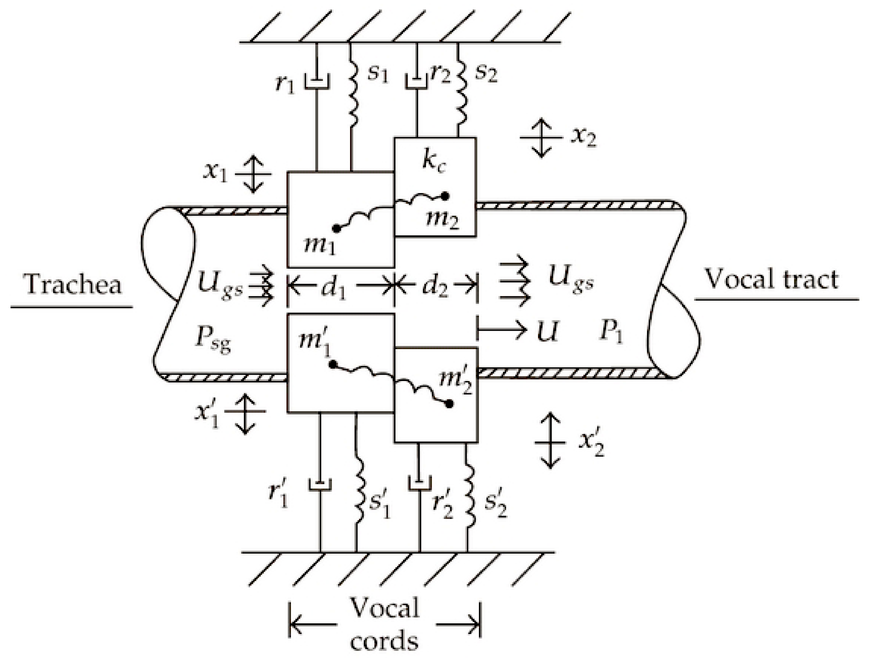


Figure 2: The two-mass model as originally proposed by Ishizaka & Flanagan [1]; figure from [5]. The m_1, m_2 and m_1', m_2' pairs correspond to the vocal folds in fig. 1; the entire mass-spring system together comprises the glottis.

Model Parameters

Two masses, m_1 and m_2, are modeled with restitution springs s_1 and s_2 and a coupling spring with linear spring constant k_c (see fig. 2). Although a simplified model, these masses roughly correspond to the two layers of tissue comprising the vocal folds, the body and cover. The body is comprised of muscle and deep layers of ligament, making up the bulk of the vocal fold; this corresponds to the larger m_1. The cover consists of more superficial tissues around the body, corresponding to m_2. d_1 and d_2 are the lengths of m_1 and m_2; we also define glottal width (in the plane normal to d) l. Rest positions of the masses are x_{10} and x_{20}.

To simplify the model, I assume bilateral symmetry; that is, the left and right systems (m_1 and m_2 vs. m_1' and m_2') are identical, oscillating in phase with each other. Analysis of asymmetrical systems is instructive in considering vocal defects such as vocal paralysis; however, such analysis is beyond the scope of this project.

Equations of Motion

The equation of motion for each mass takes the following form:

$$m_i\ddot{x}_i + b_i(x_i,\dot{x}_i) + s_i(x_i) + k_c(x_i - x_j) = F_i \tag{1}$$

where $i, j = 1, 2$; x_i is the displacement of the mass from its rest position x_{i0}; b_i is a damping term, which may be linear or nonlinear; s_i is the restitution spring force; k_c is the coupling spring constant; and F_i is the driving force acting on the mass due to glottal pressure.

For simplicity, assume a linear restoring force s_i, which includes a linear term dealing with collision of the vocal folds:

$$s_i(x_1) = \begin{cases} k_i x_i, & x_i \geqslant -x_{i0} \\ k_i x_i + h_i(x_i + x_{i0}), & \text{otherwise} \end{cases} \tag{2}$$

where h_i is an increased stiffness coefficient; and a linear damping term $b_i(x_i, \dot{x}_i) = r_i x_i$.

Setting $y_i = \dot{x}_i$, we obtain a system of first-order differential equations:

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = \frac{1}{m_1}(-r_1x_1 - s_1(x_1) - k_c(x_1 - x_2) + F_1) \\ \dot{x}_2 = y_2 \\ \dot{y}_2 = \frac{1}{m_2}(-r_2x_2 - s_2(x_2) - k_c(x_2 - x_1) + F_2) \end{cases} \tag{3}$$

Following Lucero [2], I assume a simplified pressure flow, with a pressure loss over the glottis caused by the vena contracta at the glottal inlet. The driving force F_i is caused by the subglottal pressure P_s acting on the area $a_i = ld_i$ of the masses and is given by:

$$F_1 = \begin{cases} ld_1P_sf_p, & x_1 > -x_{10}, x_2 > -x_{20} \\ ld_1P_s, & \text{otherwise} \end{cases} \tag{4}$$

$$F_2 = \begin{cases} ld_2P_s, & x_1 > -x_{10}, x_2 \leqslant -x_{20} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

The pressure loss term f_p is given by:

$$f_p = \frac{(x_1 + x_{10})^2 - (x_2 + x_{20})^2}{(x_1 + x_{10})^2 + \kappa(x_2 + x_{20})^2} \tag{6}$$

where $\kappa = 0.37$ is the pressure loss factor calculated in [1].

Equilibria

Equilibrium positions are obtained by setting $\dot{x}_i = \dot{y}_i = 0$:

$$\begin{cases} y_1 = 0 \\ F_1 = r_1x_1 + s_1(x_1) + k_c(x_1 - x_2) \\ y_2 = 0 \\ F_2 = r_2x_2 + s_2(x_2) + k_c(x_2 - x_1) \end{cases} \tag{7}$$

Considering equilibria in the open glottis; that is, $x_1 > -x_{10}$ and $x2 > -x_{10}$, we obtain expressions for equilibria:

$$\begin{aligned} k_1x_1^* + k_c(x_1^* - x_2^*) &= lg_1P_sf_p \\ k_2x_2^* + k_c(x_2^* - x_1^*) &= 0 \end{aligned} \tag{8}$$

where x_1^*, x_2^* denote equilibrium positions. Clearly,

$$x_2^* = \frac{k_c}{k_c + k_2}x_1^* \tag{9}$$

Scaling the equations in (8) following Lucero [2]:

$$\begin{aligned} \alpha &= \tfrac{k_c}{k_c+k_2}, \\ j_i &= 1 + x_i/x_{i0}, \text{and} \\ p_s &= \tfrac{ld_1P_s}{k_1x_{10}} \end{aligned} \tag{10}$$

and assuming the simplified case of a rectangular glottis; that is, $x_{10} = x_{20}$, we obtain:

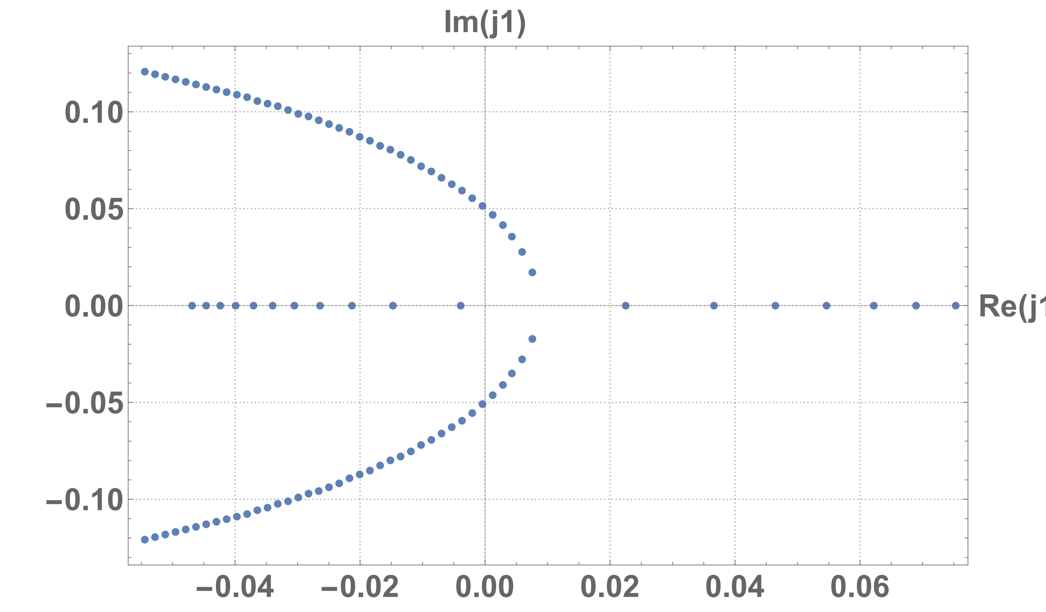
$$j_1^* = 1 + \frac{p_s}{1 + \alpha k_2/k_1} * \frac{j_1^{*2} - j_2^{*2}}{j_1^{*2} + \kappa j_2^{*2}} \tag{11}$$

and

$$j_2^* = \alpha(j_1^* - 1) + 1 \tag{12}$$

Given $x_2^* = \alpha x_1^*$ from (8), we obtain the solution $j_1^* = j_2^* = 1$, corresponding to the trivial solution $x_1 = x_2 = 0$; and two other solutions given by substituting (9) into (11).

I use experimentally determined parameters given in [3] to determine the other two equilibrium solutions j_1^* in terms of subglottal pressure p_s. It is instructive to graphically examine numerical solutions, plotted parametrically for p_s:



Bifurcation Analysis

For small p_s, $\text{Im}(j_1^*)$ is either negative or positive. The transition between negative and positive imaginary parts occurs at approximately $p_s \simeq 0.397$. For large p_s, $\text{Im}(j_1^*)$=0, and $\text{Re}(j_1^*)$ is either negative or positive. As we are restricted to the open-glottis condition, $j_1 < 0$ is not a valid solution; by (10), this would correspond to $x_1 < -x_{10}$, the closed-glottis condition. However, for $\text{Re}(j_1^*) > 0$, we have $x_1 > -x_{10}$, which is allowed under our constraints; this suggests the presence of another equilibrium in the open-glottis regime. For larger values of p_s, this second equilibrium coincides with the rest position $j_1^* = 1$, creating a bifurcation. Using the shooting method to numerically solve for p_s, we find that this bifurcation occurs at $p_s \simeq 3.04$.

Discussion

The two-mass vocal fold model has been studied with various iterations of parameters, including linear and nonlinear restoring and damping forces, symmetric and asymmetric conditions, and various assumptions regarding vocal tract geometry and airflow. In particular, bifurcations in the open-glottis regime associated with subglottal pressure have been shown to be Hopf bifurcations, giving rise to phonation through the creation of steady limit cycles [3, 5, 6, 7]. Such a bifurcation indicates the existence of a minimum subglottal pressure P_s required to enter self-sustained oscillation, making sound production possible.

Although greatly simplified, my model recovered three equilibrium solutions, at least two of which coincide to form a bifurcation associated with phonation onset. Future plans include further stability analysis, treatment of more realistic, nonlinear spring and damping forces, and a more thorough treatment of the hydrodynamics of the pressure flow.

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