Solid-Fuel Rocket Dynamics: Launch and Orbital Mechanics

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Introduction:

In Math 46: Introduction to Applied Mathematics, course material included various applications of differential equations to physical and chemical processes. In this final project, we seek to apply some of our newfound knowledge of these techniques to the physics of a rocket launch and orbit.

Our model involves equations relating to fuel consumption, rocket trajectory, and eventual satellite orbit. First, we examine the process of fuel consumption within the engine, which gives rise to a differential equation describing the change in rocket mass over time. This equation becomes a parameter for thrust, which determines the rocket's upward motion. As the rocket moves away from earth's surface, drag and gravitational force also become factors in the system of equations describing its trajectory. Finally, we examine the physics of a satellite in orbit as a function of initial conditions derived from the rocket's flight. In the process of developing the model, we use various approximation techniques to produce analytically solvable equations. We then construct a final model in MATLAB, where we can view our simulations and the effects of changing parameters. We also use the model to "slingshot" the theoretical rocket around a body such as the moon.

Solid Fuel Burning and Grain Geometry:

The amount of thrust provided by the engine throughout the burn is critical. In solid-fuel rockets, the main way to achieve a desired thrust curve is by changing the geometry of the fuel within the rocket. Specifically, the instantaneous surface area of the propellant is important to consider. A rocket is propelled forward by forcing a stream of mass (exhaust) backwards. In a rocket burning solid fuel, this mass flow rate is described by the following equation:

$$\frac{dm}{dt} = \rho * b * A(t)$$

where ${}^{dm}/_{dt}$ is the mass flow rate, ρ is the density of the fuel, b is the linear burn rate of the fuel, and $A(t)$ is the instantaneous surface area of the fuel (the part burning) at time t.

The above equation applies when modeling the the "quasi-steady state" portion of the burn, which excludes ignition and burnout (Sullwald 7). It is reasonable to assume that the density of the fuel and linear burn rate will be constant throughout this time. The burning surface area will certainly evolve over time, and therefore the shape of the fuel, or grain geometry, is important in determining the mass flow rate, and then the mass of the rocket itself. Here we track how the burning surface evolves by assuming that burning will occur in a direction normal to the burning surface, and at rate b as stated above. The cross sections of three possible fuel arrangements are shown below.

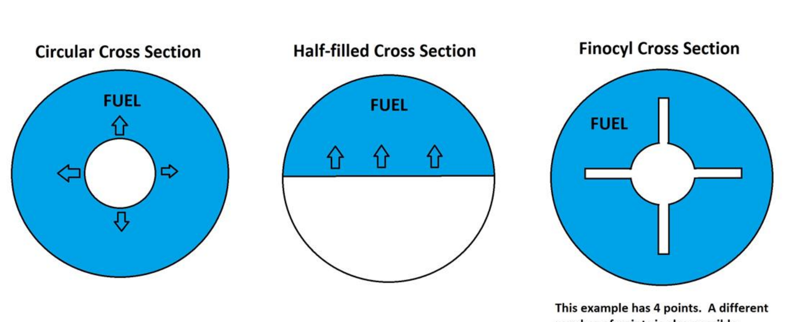
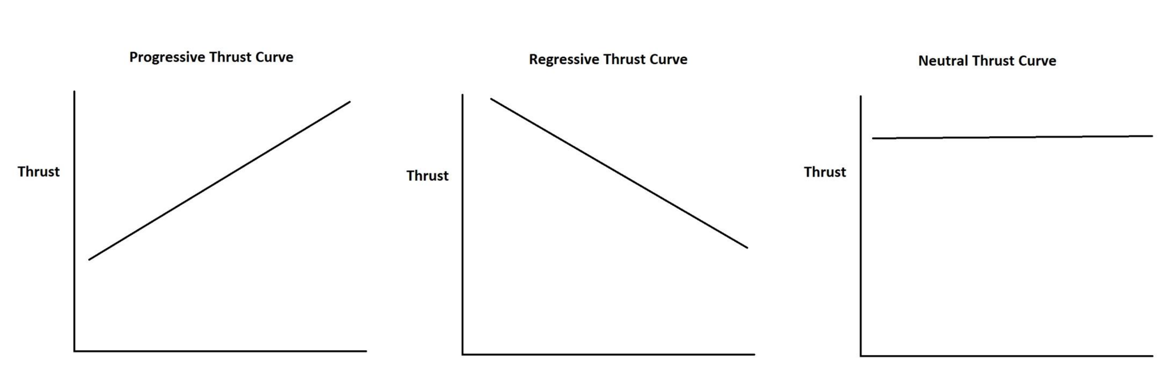


Figure 1: Three potential cross sections showing different grain geometries. Left to right: circular bore, semi-circular structure, four-point finocyl.

In practice, fine-tuning of the grain geometry is used to achieve specific thrust levels needed for different rocket missions. Progressive, regressive, and neutral thrust curves are basic ways to classify the thrust of a rocket over time. Diagrams for these concepts are shown below. The circular, half-filled, and finocyl cross section could be used to provide progressive, regressive, and neutral thrust, respectively.



The following chart shows some implications of the different grain structures on mass flow rate as well as the mass of the rocket throughout the burn. In each case, an evolving surface area leads to a mass flow rate equation. Integration can be used to find the total mass of exhaust at a certain time, and that mass equation can be used to determine the mass of the rocket at any given time.

Basic Circular Bore:

Surface Area:

$$A(t) = 2\pi * r(t) * h$$
$$r(t) = r_0 + bt$$

Mass Flow Rate:

$$\frac{dm}{dt} = \rho * b * 2\pi(r_0 + bt) * h$$

Mass of fuel burned over time:

$$m(t) = 2\pi\rho bh * \left(r_0 t + \frac{bt^2}{2}\right)$$

Semi-circular cross section:

Surface Area:

$$A(t) = a(t) * h$$
$$a(t) = 2\sqrt{r^2 - (bt)^2}$$

Mass Flow Rate:

$$\frac{dm}{dt} = \rho * b * 2\sqrt{r^2 - (bt)^2} * h$$

Mass of fuel burned over time:

$$m(t) = 2\rho bh \int \sqrt{r^2 - (bt)^2}\,dt$$

Finocyl:

Due to its complex shape, the finocyl requires some simplifying assumptions to be made regarding the evolution of the burning surface. For the purposes of this project, we will make a selective burning assumption which results in piecewise functions.

Surface Area:

$$A(t) = \begin{cases} \left(2\pi * r(t) + 2n * d(t)\right) * h & t < \frac{r_0 + d_0}{b} \\ 2\pi * r(t) * h & t \geq \frac{r_0 + d_0}{b} \end{cases}$$

$$r(t) = r_0 + bt \qquad d(t) = d_0 - bt$$

Mass Flow Rate:

$$\frac{dm}{dt} = \rho * b * A(t)$$

Mass of fuel burned over time:

$$m(t) = pb \int A(t)dt$$

Rocket Mass:

In all cases, the mass M of the rocket is

$$M(t) = M_0 - m(t)$$

$r_0 = initial\ radius,\quad b = burn\ rate,\quad h = height\ of\ fuel\ cylinder,\quad \rho = fuel\ density$
$d = length\ of\ fin,\quad d_0 = initial\ fin\ length,\quad n = number\ of\ fins,$
$M_0 = initial\ mass\ of\ rocket, including\ fuel$

Rocket Equation and Flight Path:

After determining the mass flow rate, dm/dt, and the mass of fuel burned over time, m(t), for the rocket, the mass equations become parameters for a model of the rocket's flight. In its most basic form, the rocket equation describes the mass of the rocket times its acceleration in terms of three forces: thrust, gravity, and drag. Thrust is the force with which the expelled mass propels the rocket upward, and it is expressed as the product of the velocity of the mass being expelled and dm/dt. Gravity, a function of the rocket's distance from the center of the earth, and drag, a function of its velocity, work in the opposite direction to slow the rocket as it moves upward.

In order to analyze the rocket's flight, we can use either a full equation incorporating all of these forces in their most complex forms, or simplified versions to solve analytically. The following are the full versions and simplifying assumptions that we use for each of the three forces:

Rocket Equation:

$$m(t)x'' = F_{thrust} - F_{gravity} - F_{drag}$$

Thrust

$$m(t)\frac{dv}{dt} = -V\frac{dm}{dt}$$

$$\frac{dv}{dt} = -V\frac{1}{m(t)}\frac{dm}{dt}$$

$$v = -Vln\left(\frac{m}{m_0}\right) = Vln\left(\frac{m_0}{m_f}\right)$$

Gravity

Simple form: $F_g = m(t)g$

Complex Form: $F_g = \frac{GMm(t)}{R^2}$

Drag

Simple form: $F_D = 0$

Complex Form: $F_D = \beta V = \beta x'$

$$m(t)x'' = V\frac{dm}{dt} - \frac{GMm(t)}{x^2} - \beta x'$$

As a simple, analytically solvable form, we assume that there is no drag on the rocket, and that the force of gravity is a constant, g, rather than a function of the rocket's distance from the earth. These simplifications allow us to express force as (thrust - gravity) and integrate twice from the initial to the final mass to solve for x(t):

1. Simple Form to Solve Analytically

$$m(t)\frac{dv}{dt} = V\frac{dm}{dt} - gm(t)$$
$$\int dv = \int \frac{V}{m(t)}dm - gdt$$

$$v(t) = Vln\left(\frac{m_0}{m_f}\right) - gt + v_0$$
$$\frac{dx}{dt} = Vln\left(\frac{m_0}{m_f}\right) - gt + v_0$$

$$x(t) = \left[Vln\left(\frac{m_0}{m_f}\right) + v_0\right]t - \frac{1}{2}gt^2 + x_0$$

Saturn V Rocket. Source: Kevin Boudreaux, Angelo State University.

If we express all parameters in their most complex form, using variable gravity and both linear and quadratic drag, we produce the following equation, which can be solved via MatLab simulation:

2. Complex Form to Solve in MatLab

$$m(t)x'' = V\frac{dm}{dt} - \frac{GMm(t)}{(x+R)^2} - \beta x' - \beta_2(x')^2$$

Finally, by making a few adjustments, we can produce an approximation to the rocket equation using perturbation methods:

3. Perturbation Approximation

$$m(t)x'' = V\frac{dm}{dt} - gm(t) - \beta x'$$

We assume $\frac{dm}{dt}$ to be a constant, c. Then, in order for our perturbation methods to apply, we also must treat the mass of the rocket as a constant $m(t) = M$, to obtain an approximate solution.

$$Mx'' = Vc - gM - \beta x'$$
$$x'' = \frac{Vc}{M} - g - \frac{\beta}{M}x'$$
$$x'' + \varepsilon x' - \left(\frac{Vc}{M} - g\right) = 0$$

Let $x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

Initial conditions: $x(0) = 0, x'(0) = V_0$

$$(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + \varepsilon(x_0' + \varepsilon x_1' + \varepsilon^2 x_2' + \cdots) - \left(\frac{Vc}{M} - g\right) = 0$$

ε^0: $x'' - \left(\frac{Vc}{M} - g\right) = 0 \qquad x_0(t) = \frac{1}{2}\left(\frac{Vc}{M} - g\right)t^2 + V_0 t$

ε^1: $x_1'' + x_0' = 0 \qquad x_1(t) = -\frac{1}{6}\left(\frac{Vc}{M} - g\right)t^3 - \frac{1}{2}V_0 t^2$

Then $x(t) = x_0 + \varepsilon x_1$ and so forth, to obtain closer approximations of the rocket's motion, and one could use one or both of the following cases for M:

Case 1: $M = initial\ mass\ of\ rocket$

Case 2: $M = empty\ mass\ of\ rocket$

Simulation Details:

The full system of differential equations—from launch to orbit—was solved numerically with a system of first-order ODE's using MATLAB's 'ode45' solver. The program incorporates linear and quadratic drag forces, gravitation due to the Earth, gravitation due to the moon, thrust, and variable rocket mass.

$$\ddot{x} = \frac{-MGx}{(x^2+y^2)^{3/2}} + \frac{M_{moon}G(R-x)}{((R-x)^2+y^2)^{3/2}} + F^x_{thrust} + F^x_{drag}$$
$$\ddot{y} = \frac{-Mgy}{(x^2+y^2)^{3/2}} + \frac{M_{moon}Gy}{((R-x)^2+y^2)^{3/2}} + F^y_{thrust} + F^y_{drag}$$

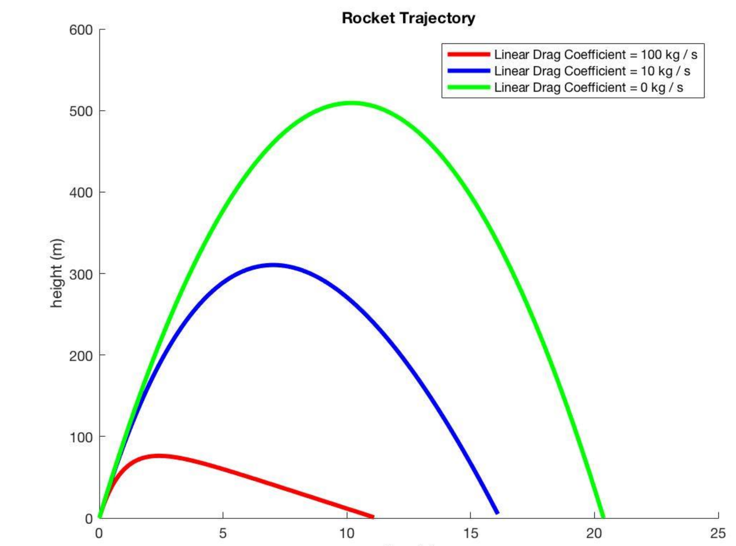
We made the following simplifying assumptions: The Earth and the moon are fixed in space (with the moon a distance R along the x-axis from the Earth), and the atmosphere is of constant density, which goes abruptly to zero at whatever height the rocket's engines cut out.

Our program takes as input symbolic expressions for gravitation, thrust, and drag. It converts them to a matrix A such that **y'**= A**y**, where

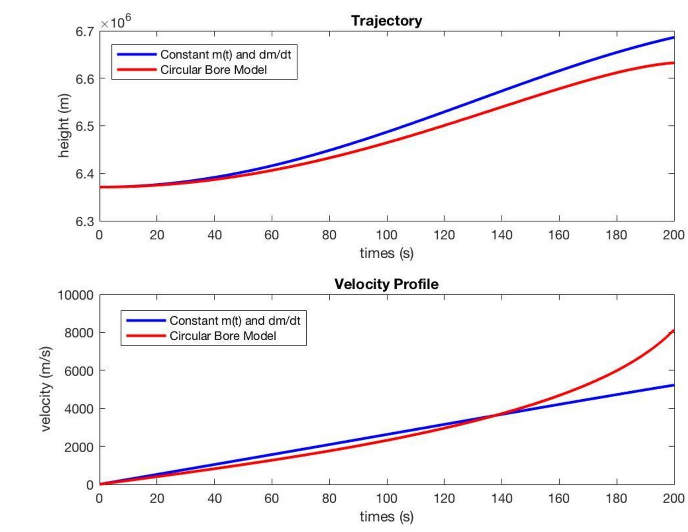
$$\mathbf{y} = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

This matrix equation is solved with ode45. A few examples of the program's capabilities are shown on this poster.

Simulating Launch Parameters:



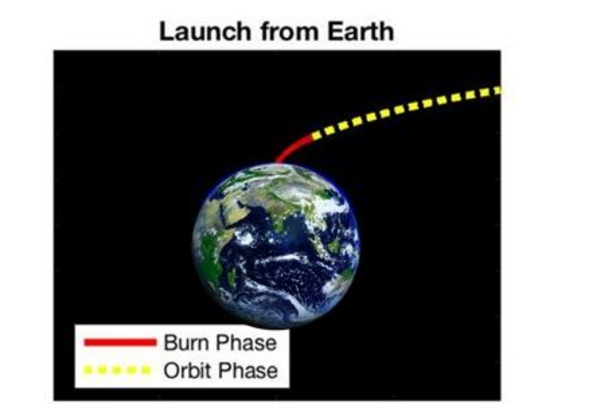
A demonstration of the dramatic influence of linear drag parameters on flight trajectory, holding other parameters constant with no thrust.

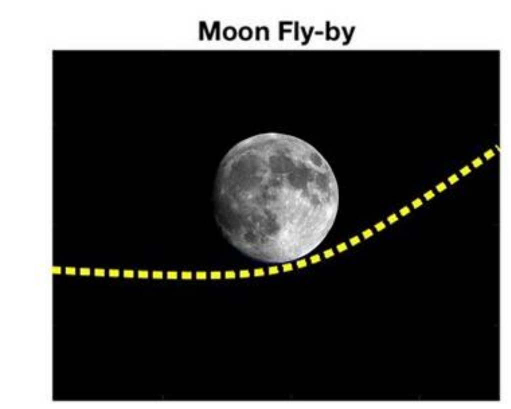


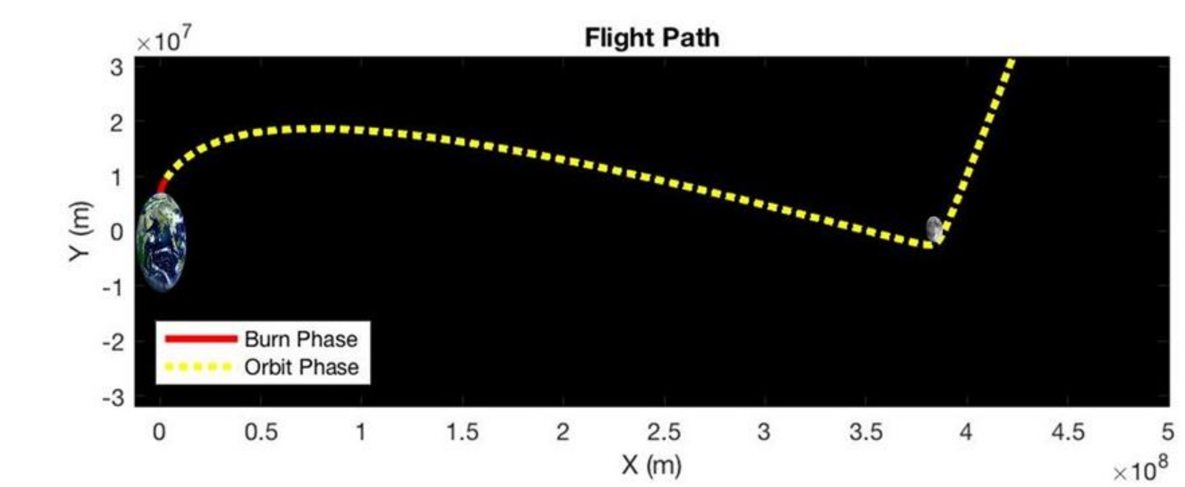
Flight profiles comparing an imaginary rocket with constant m(t) and dm/dt against the behavior of the circular bore fuel geometry.

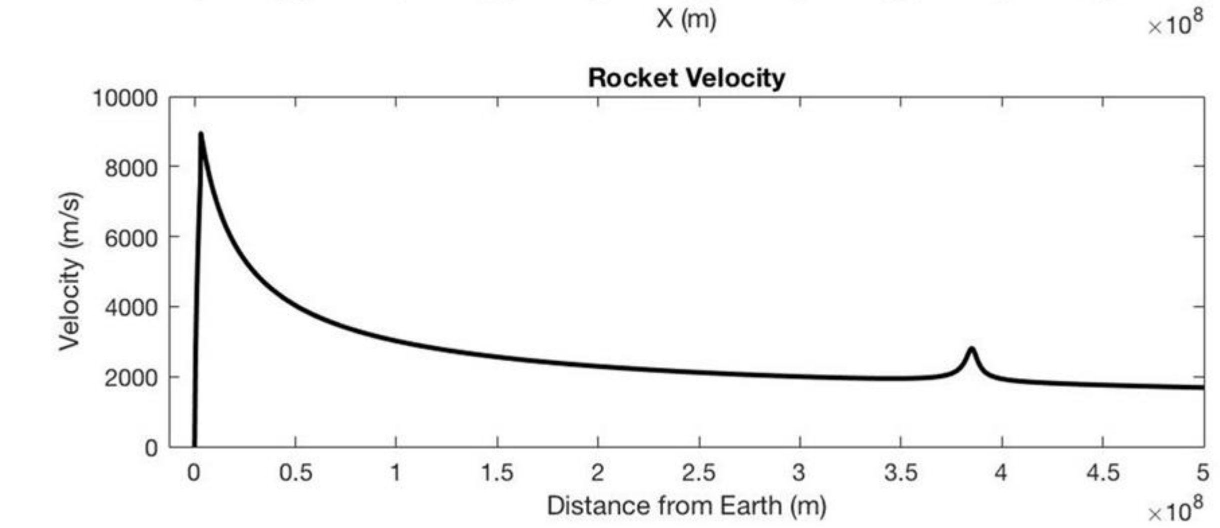
Orbital Simulation and Mechanics:

The various launch parameters – fuel geometry, drag forces, initial and final mass – produce a launch trajectory that terminates when the satellite is ejected from the rocket. At this point, we simulate the flight path using the gravitational fields of both the Earth and the moon. Below is shown one such trajectory, which demonstrates the use of a celestial body as a gravitational "slingshot" to shape the path of the satellite.









Conclusion:

A combination of analytical approximations and numerical solutions give a complete picture of the behavior of a solid fuel rocket from launch to spaceflight. In particular, we consider the impact of fuel grain geometry, drag, and gravitation due to the earth and the moon. On the whole, careful designation of parameters produces highly plausible flight paths for a range of scenarios. There are several promising avenues of extension for this work. Our model could consider multi-stage rockets or mid-orbit burns. We could also create a more sophisticated model of drag in which atmospheric density falls off as a function of altitude. In terms of orbital flight, we could improve accuracy by incorporating the moon's orbit around the earth instead of assuming it to be stationary, and we could also generalize to more bodies in the solar system.

References:

Peraire, J. and S. Widnall. "Lecture 14 – Variable Mass Systems: The Rocket Equation" MIT, Fall 2008.

Sullwald, Wichard. "Grain Regression Analysis." University of Stellenbosch, 2009.