Hamiltonian Dynamics of Love Triangles

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INTRODUCTION

An obvious difficulty in any mathematical analysis of the dynamics of romance is defining what is meant by love and quantifying it meaningfully (Sternberg & Barnes 1988). One such model, proposed by Sprott (2004) assumes a simple linear construction where the growth rate of individual A's love for individual B (dA/dt), depends on the overall "quantity" of love individual A harbors for B at time t (A(t)) and the overall "quantity" of love individual B harbors for A at time t (B(t)), and the same for dB/dt. Sprott then goes on to add additional terms in the model to account for love triangles and the presence of nonlinearities.

A novel approach in analyzing the dynamics of love triangles involves the use of the analogous three-body problem in classical mechanics, which takes an initial set of data that specifies the positions, masses, and velocities of three bodies for some particular point in time and subsequently determines the motions of the three bodies, in accordance with Newton's laws of motion and of universal gravitation. Alternatively, the three body problem can also be approached using energy considerations, via a Hamiltonian approach.

METHODS: MODEL A

The love situation: three individuals I_1 , I_2 and I_3 are involved in a love triangle: I_1 and I_3 are competing rivals and in love with I_2. The degree of freedom at play in this model is the one-dimensional **emotional distance** as a function of time x=x(t). Each individual I_k is characterized by an emotional inertia m_k that scales with resistance to change in emotional distance $x_k(t)$. Love (attraction) between two individuals I_i and I_j results from a low $|x_i(t)-x_j(t)|$. In this mathematical model, the emotional distances will undergo time fluctuations as a result of interaction models, set by interaction potentials. This energetics approach can be fully fleshed out using the Hamiltonian formalism, after developing the kinetic and potential energy terms. Qualitatively, any candidate model will have to satisfy two love constraints: I_1 and I_3 must be *attracted* to I_2, and *repel* each other.

The first model (A): Spring-force attraction and Gaussian repulsion. The spring-force models periodically rebouncing relationships (with consecutive infatuation and separation cycles), while the repulsion models jealousy that peaks as the two suitors I_1 and I_3 approach their common love interest. Mathematically, the attraction potential will be quadratic, while the repulsion potential will be a Gaussian:

$$U_1 = \frac{1}{2}k_{12}(x_1-x_2)^2 + \alpha_{13}e^{-\beta_{13}(x_3-x_1)^2}$$

$$U_2 = \frac{1}{2}k_{12}(x_1-x_2)^2 + \frac{1}{2}k_{23}(x_2-x_3)^2$$

$$U_3 = \frac{1}{2}k_{32}(x_3-x_2)^2 + \alpha_{13}e^{-\beta_{13}(x_3-x_1)^2}$$

Incorporating the kinetic energies (in terms of momenta p_k , each representing the emotional impulsivities of the individual I_k) to these potential terms yields the effective Hamiltonian for this love model:

$$\mathcal{H}_A = \frac{1}{2}\Big(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}+\frac{p_3^2}{m_3}\Big) + \Big(k_{12}(x_1-x_2)^2 + k_{23}(x_2-x_3)^2 + 2\alpha_{13}e^{-\beta_{13}(x_1-x_3)^2}\Big)$$

Given this Hamiltonian, the equations of motion, using x_k as coordinates and p_k as momenta:

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i};\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i} \rightarrow \dot{x}_i = \frac{p_i}{m_i} \text{ and}$$

$$\dot{p}_1 = -2k_{12}(x_1-x_2) + 4\alpha_{13}\beta_{13}e^{-\beta_{13}(x_1-x_3)^2}$$

$$\dot{p}_2 = 2k_{12}(x_1-x_2) + 2k_{23}(x_3-x_2)$$

$$\dot{p}_3 = 2k_{23}(x_2-x_3) - 4\alpha_{13}\beta_{13}e^{-\beta_{13}(x_1-x_3)^2}$$

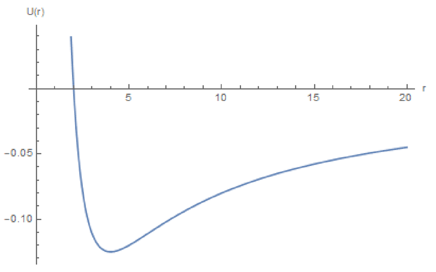
METHODS: MODEL B

The second model (B): London potential attraction and Coulombian repulsion. The London potential models tentative invitation at a distance (attractive) and a "rejection radius" (sudden repulsion), as a non-trivial, unrequited love situation. The Coulombian repulsion just models the repulsion between the two suitors, with their inertias acting like repulsing charges. The potential energies are thus:

$$U_1 = \Big(\frac{\alpha_{12}}{(x_1-x_2)^2} - \frac{\beta_{12}}{x_1-x_2}\Big) + \frac{q_1q_3}{x_1-x_3}$$

$$U_3 = \Big(\frac{\alpha_{32}}{(x_3-x_2)^2} - \frac{\beta_{32}}{x_3-x_2}\Big) + \frac{q_1q_3}{x_1-x_3}$$

$$U_2 = \Big(\frac{\alpha_{12}}{(x_1-x_2)^2} - \frac{\beta_{12}}{x_1-x_2}\Big) + \Big(\frac{\alpha_{32}}{(x_3-x_2)^2} - \frac{\beta_{32}}{x_3-x_2}\Big)$$



$U(r) = \frac{\alpha}{r^2} - \frac{\beta}{r}$ for $\alpha = 2$ and $\beta = 1$

Yielding an effective Hamiltonian:

$$\mathcal{H}_B = \frac{1}{2}\Big(\frac{p_1^2}{q_1}+\frac{p_2^2}{q_2}+\frac{p_3^2}{q_3}\Big) + 2\Big(\frac{\alpha_{12}}{(x_2-x_1)^2} + \frac{\alpha_{23}}{(x_2-x_3)^2} - \frac{\beta_{12}}{(x_1-x_2)} - \frac{\beta_{23}}{(x_3-x_2)} + \frac{q_1q_3}{x_1-x_3}\Big)$$

Solving for the equations of motion:

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i};\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i} \rightarrow \dot{x}_i = \frac{p_i}{q_i} \text{ and}$$

$$\dot{p}_1 = \frac{4\alpha_{12}}{(x_1-x_2)^3} - \frac{2\beta_{12}}{(x_1-x_2)^2} + \frac{2q_1q_3}{(x_1-x_3)^2}$$

$$\dot{p}_2 = -\frac{4\alpha_{12}}{(x_1-x_2)^3} + \frac{2\beta_{12}}{(x_1-x_2)^2} + \frac{4\alpha_{23}}{(x_2-x_3)^3} + \frac{2\beta_{23}}{(x_2-x_3)^2}$$

$$\dot{p}_3 = -\frac{4\alpha_{23}}{(x_2-x_3)^3} - \frac{2\beta_{23}}{(x_2-x_3)^2} - \frac{2q_1q_3}{(x_1-x_3)^2}$$

RESULTS

Model A:

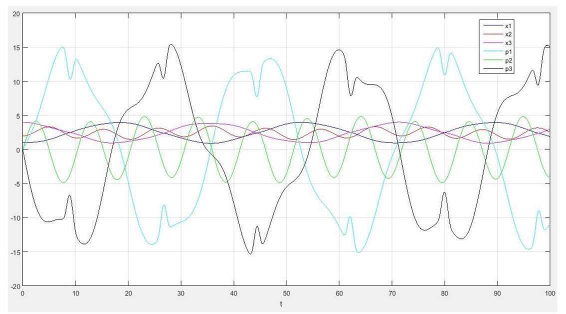


Fig 1: Case (i)

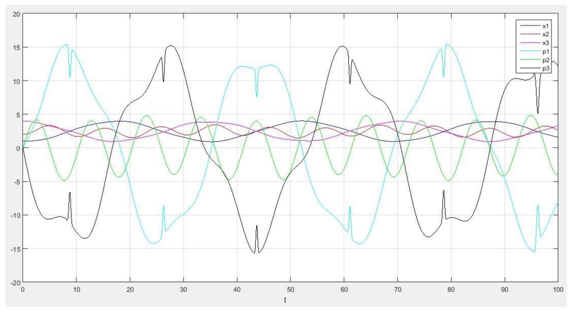


Fig 2: Case (ii)

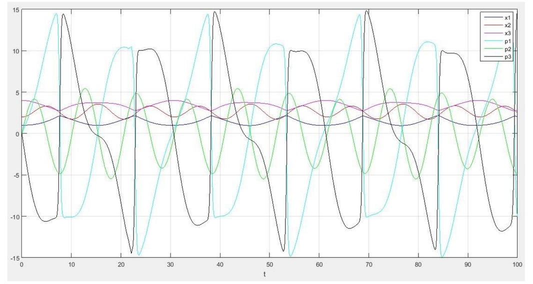


Fig 3a: Case (iii)

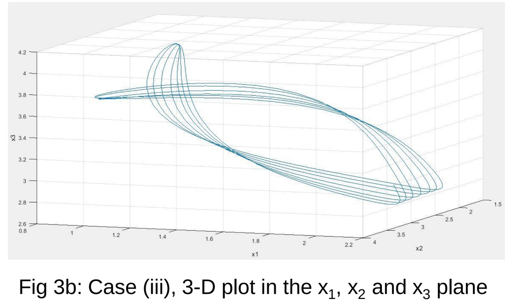


Fig 3b: Case (iii), 3-D plot in the x_1, x_2 and x_3 plane

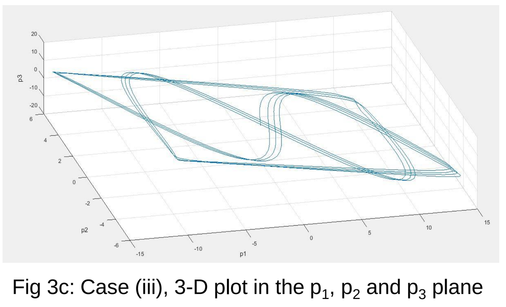


Fig 3c: Case (iii), 3-D plot in the p_1, p_2 and p_3 plane

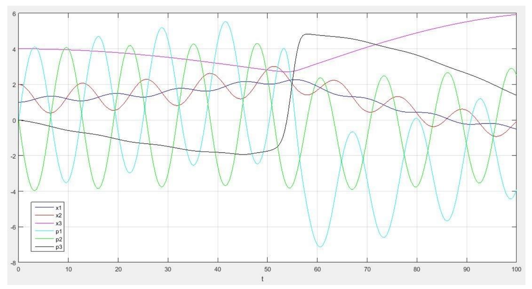


Fig 4: Case (iv)

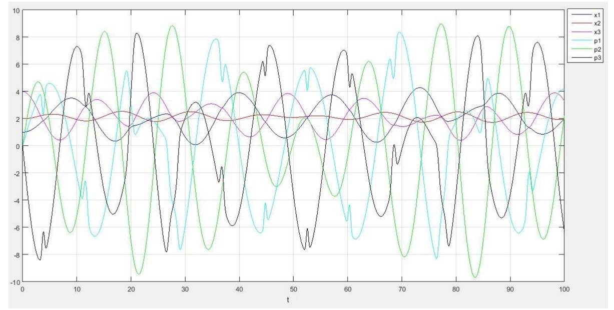


Fig 5: Case (v)

Model B:

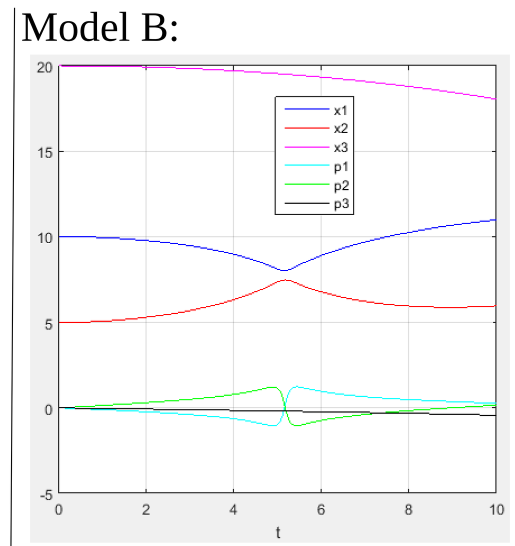


Fig 1: Before model breakdown (t=0 to t=10)

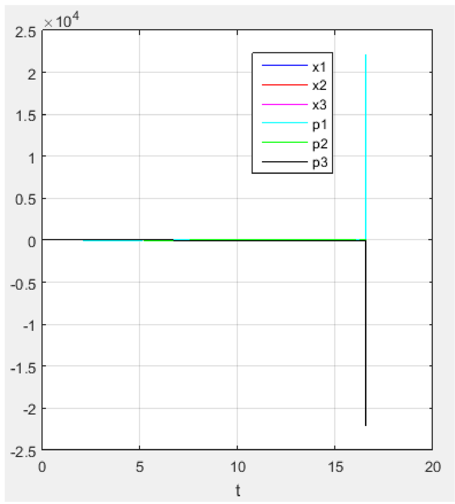


Fig 2: Illustration of model breakdown at t~16

DISCUSSIONS

Model A:

As aforementioned, the parameters m_k correspond to emotional inertia, which is essentially a resistance of individuals to change their state in the emotional space. α_{13} represents the intensity of jealousy between individuals I_2 and I_3, while β_{13} represents the sharpness of this jealousy, henceforth termed the "vitriol constant". k_{12} and k_{23}, analogous to spring constants in a mechanical model, scale to the emotional impetuousness between two individuals, termed the "infidelity constant". Since individuals I_1 and I_3 are both in love with individual I_2, we make the assumption that they have identical emotional traits: specifically, their emotional inertias m_1 and m_3 are identical. Since only the relative emotional distance between individuals matters for our model, we chose initial conditions x_1=1, x_2=2 and x_3=4 for all the cases.

Case (i) illustrates the dynamics of the system for representative values of m_1=50, m_2=10, m_3=50, α_{13}=1, β_{13}=10, k_{12}=0.7 and k_{23}=1. I_1 and I_3 start off nearing I_2, while I_2 follows oscillations owing to an effective I_2 constant from the two attractions it undergoes. When I_1 and I_3 emotionally collide, their impulsivities spike and dip as they rush to avoid each other (collisions). The dip results from the impulsivities decreasing in magnitude due to the mutual repulsion as I_1 and I_3 approach each other; the spike is a resurge of impulsivity as the individuals traverse each other in emotional space.

Case (ii): Identical to case (i) but with a large vitriol constant (β_{13}=100). This results in sharper impulsivity spikes in the plots.

Case (iii): Identical to case (i) but with a large jealousy intensity (repulsion) between individuals I_1 and I_3 (α_{13}=100). The high repulsion between I_1 and I_3 result in them veering away as they approach each other while approaching I2. Unlike case (i), there are no collisions in this case between I_1 and I_3. Additional 3-D plots are included for this case, for x_1, x_2 and x_3, and for p_1, p_2 and p_3, because of their interesting geometries.

Case (iv): Identical to case (i) but with a larger magnitude difference in the infidelity constant (k_{12}=1, k_{23}=0.01). Since k is inversely to proportional to the time period of oscillations, the time period of both emotional distance and impulsivity oscillations greatly increase for I_3 relative to I_1 and I_2. This corresponds to a longer relationship span between I_3 and I_2.

Case (v): Identical to case (i) but with I_1 and I_3 having smaller emotional inertias than I_2 (q_1=10, q_2 = 50, q_3 = 10). This plot is included for the sake of completeness while iterating through different parameter values.

Model B:

This model illustrates consecutive approach-and-repulsion cycles, with I_1 and I_3 strongly repelling each other as they approach. This model leads to singularities and discontinuous derivatives, as shown in the second plot and hence does not lend itself to conclusive Hamiltonian analysis. Consequently, this model was not analyzed further.

CONCLUSIONS

We have studied different functional forms for the interactions in our love triangle model, and encountered some of the main challenges inherent in applied mathematics: connecting the mathematical formalism with the tangible elements of the applied situation at hand. Some of our results were non-trivial and could potentially be tied to plausible human relationship occurrences.

While the analysis only covered two Hamiltonian ansätze, we hope that this can be the start to further studies of love-triangles (or even love-polygons) with more parameters and richer interaction models.

We would like to thank Professor Malik for giving us the opportunity to conduct this project and advising us throughout its course.