Chaos: Lorenz Equations, Strange Attractors and Fractals

Lorenz Equations

Chaos was first discovered by Lorenz in 1963 when he studied the 3-D equations.

$$\begin{cases} x' = \sigma(y-x) \\ y' = rx - y - xz \\ z' = xy - bz \end{cases} \qquad (\sigma,\ r,\ b \text{ are constant parameters})$$

which was a drastically simplified model of convection rolls in the atmosphere. The same equations also arise in models of lasers and dynamos.

Lorenz equations form a 3-D autonomous system. We know that for a 2-D autonomous system, the trajectories in the phase plane are always attracted to a stable critical point, or a periodic closed orbit, or infinity, and things seem simple enough.

But what about 3-D autonomous systems?

We'll show that things can get crazy and Lorenz equations are the most famous example.

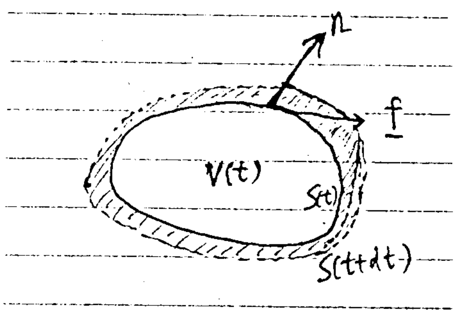
1. Simple Properties of the Lorenz Equation

   (1) Nonlinearity

   (2) Symmetry: $(x, y) \to (-x, -y)$, the equations remain the same.
   So if $(x(t), y(t), z(t))$ is a solution,
   So is $(-x(t), -y(t), z(t))$.

   (3) Volume Contraction

   For a general 3-D system $\underline{x}' = \underline{f}(\underline{x})$,
   Pick an arbitrary closed surface $S(t)$ of volume $V(t)$ in phase space.
   How would the volume $V(t)$ evolve?

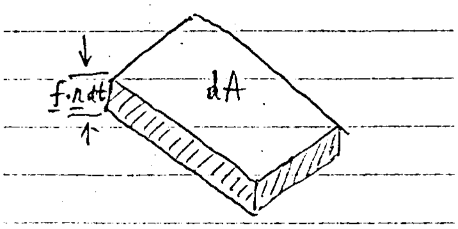


In dt time.

$$V(t+dt) = V(t) + \int_S (\underline{f} \bullet \underline{n}dt)dA$$

$$\Rightarrow \quad V'(t) = \int_S \underline{f} \bullet \underline{n}dA$$

$$= \int_V \nabla \bullet \underline{f}dV$$



For the Lorenz system,

$$\nabla \bullet \underline{f} = [\sigma(y-x)]_x + (rx - y - xz)_y + (xy - bz)_z$$

$$= -\sigma - 1 - b < 0$$

So $\quad V'(t) = -(\sigma + 1 + b)V$

$\Rightarrow \quad V(t) = V(0)e^{-(\sigma+1+b)t}$

Thus volumes in phase space shrink exponentially fast.

Hence, if we start with an enormous solid blob of initial conditions, it eventually shrinks to a limiting set of zero volume, like a balloon with the air being sucked out of it.

All trajectories starting in the blob end up somewhere in this limiting set. Later we'll see that it consists of stable critical points, limit cycles, or for some parameter values, a strange attractor.

(4) The Trajectories are Always Bounded

Consider a spherical surface $\quad S_R\colon x^2 + y^2 + (z - r - \sigma)^2 = R^2.$

If a trajectory starts on this surface, then

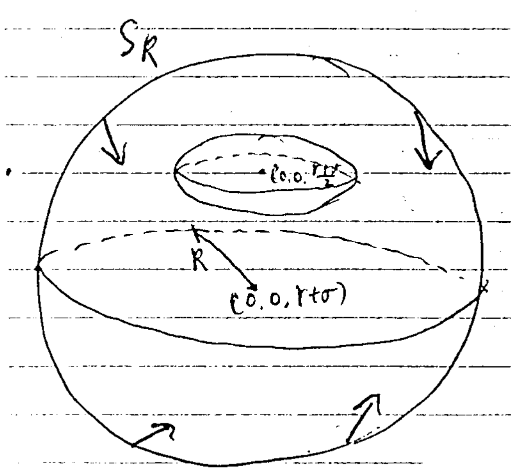
$$[x^2 + y^2 + (z - r - \sigma)^2]' = 2xx' + 2yy' + 2(z - r - \sigma)z'$$

$$= \quad \bullet\bullet\bullet \quad = -2[\sigma x^2 + y^2 + b(z - \tfrac{r+\sigma}{2})^2 - \tfrac{b(r+\sigma)^2}{4}]$$

Let us choose R big enough so that the spherical surface

$x^2 + y^2 + (z - r - \sigma)^2 = R^2 \quad$ encloses the fixed ellipsoid:

$$\sigma x^2 + y^2 + b(z - \tfrac{r+\sigma}{2})^2 = \tfrac{b(r+\sigma)^2}{4}.$$



Then on this spherical surface $\quad S_R$,

$$[x^2 + y^2 + (z - r - \sigma)^2]' < 0,$$

So trajectories starting on this surface $\quad S_R$ always come into the sphere and never go out. In other words, the sphere is a trapping region. If the initial condition is inside it, the trajectory is always confined in it. So the trajectories are always bounded and they never go to infinity.

1. Fixed Points (Critical Points).

$$\begin{cases} y-x=0 \\ rx-y-xz=0 \\ xy-bz=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \text{ and } \begin{cases} x=y=\pm(b(r-1))^{\frac{1}{2}} \\ z=r-1 \end{cases} \qquad \underline{(r \geq 1)}$$

So critical points are $(0, 0, 0)$

and when $r > 1$,

$$(\sqrt{b(r-1)}, \quad \sqrt{b(r-1}, \quad r-1) \quad \leftarrow \quad C^{+}$$

$$(-\sqrt{b(r-1)}, \quad -\sqrt{b(r-1)}, \quad r-1) \quad \leftarrow \quad C^{-}$$

2. Linear Stability of the Origin.

The linearized equations around the origin are

$$\begin{cases} x' = \sigma(y-x) \\ b' = rx-y \\ z' = -bz \end{cases}$$

So in z direction, $z \to 0$ exponentially.

In (x, y) plane,

$$\binom{x}{y}' = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \binom{x}{y}$$

$$\lambda^2 + (1+\sigma)\lambda + \sigma(1-r) = 0$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-(1+\sigma)\pm\sqrt{(\sigma-1)^2+4\sigma r}}{2}$$

$$\begin{cases} \text{when } 0 < r < 1, & \text{the two eigenvalues are both real and negative} \\ \text{when } r > 1, & \lambda_1 > 0,\ \lambda_2 < 0 \end{cases}$$

So when $0 < r < 1$, $(0, 0, 0)$ is a stable node.

when $r > 1$, $(0, 0, 0)$ is unstable.

3. Global Stability of the Origin.

Actually, when $0 < r < 1$, every trajectory is attracted to the origin as $t \to \infty$. The origin is globally stable. Hence there can be no limit cycles or chaos for $r < 1$.

To prove the global stability,

consider the function

$$V(x, y, z) = \tfrac{1}{\sigma}z^2 + y^2 + z^2$$

$$V' = \tfrac{2}{\sigma}xx' + 2yy' + 2zz' = \bullet\bullet\bullet\bullet$$

$$= 2[(r+1)xy - x^2 - y^2 - bz^2]$$

$$= -2[x - \tfrac{r+1}{2}y]^2 - 2[1 - (\tfrac{r+1}{2})^2]y^2 - 2bz^2$$

when $0 < r < 1$, $1 - (\frac{r+1}{2})^2 > 0$.

So V' is strictly negative if $(x, y, z) \neq (0, 0, 0)$.

We therefore conclude that

as $t \to \infty$, $V \to 0$. i.e. $(x, y, z) \to (0, 0, 0)$.

4. Stability of C^+ and C^- for $r > 1$.

$$\text{The Jacobian} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

At C^+: $(\sqrt{b(r-1)}, \sqrt{b(r-1}, r-1)$

$$\text{The Jacobian} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix}$$

At C^-: $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

$$\text{The Jacobian} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{bmatrix}$$

It is easy to show that, if $\sigma - b - 1 > 0$,

then C^+ and C^- are linearly stable for

$$1 < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$$

and are linearly unstable if $r > r_H$.

(If $\sigma - b - 1 \le 0$, C^+ and C^- are always linearly stable.)

At the bifurcation point $r = r_H$,

$$\lambda_{1,2} = \pm\sqrt{b(\sigma+r)}\, i, \qquad \lambda_3 = -(b+\sigma+1)$$

So we expect a Hopf bifurcation

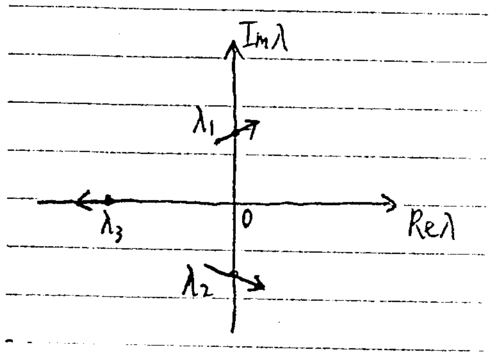
at $r = r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$.

Is it supercritical or subcritical?

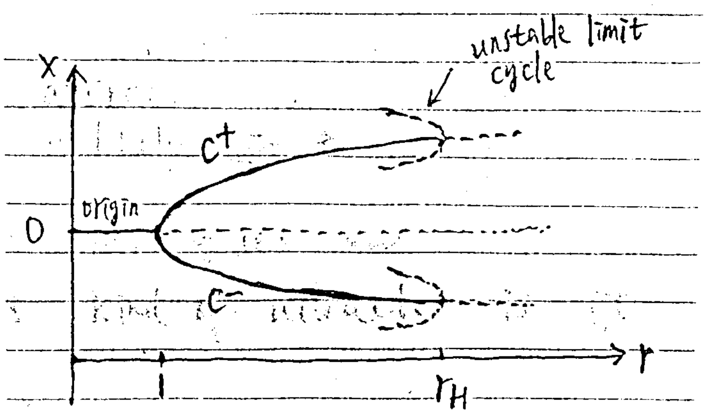
This requires a difficult calculation,

and it turns out that it is a

subcritical Hopf bifurcation.



So when r is slightly greater than r_H, there will not be small stable limit cycles around C^+ and C^-, and the trajectories are repelled out of the C^+ and C^- neighborhood and attracted to a distant attractor.



Now we summarize our results:

1. When $0 < r < 1$, the origin $(0, 0, 0)$ is globally stable. All trajectories are attracted to it.

2. At $r = 1$, a super critical pitchfork bifurcation occurs.

   Two stable critical points C^+ and C^- are created and at the same time, the origin loses its stability.

3. When $1 < r < r_H$, C^+ and C^- are stable critical points.

4. At $r = r_H$, a subcritical Hopf bifurcation occurs,

   and C^+ and C^- lose their stability.

Question: What happens when $r > r_H$?

What we know is:

1. There are no stable critical points any more.
   The origin, C^+ and C^- all become unstable.

2. There are no limit cycles around C^+ and C^- due to the subcritical Hopf bifurcation.

3. The trajectories are always bounded and can not escape to infinity.

4. Volumes contract exponentially fast. If we start with an enormous solid blob of initial conditions, it eventually shrinks to a limiting set of zero volume. This limiting set, or attractor, does not contain any critical point. It is also unlikely to contain limit cycles.

What limiting set will the trajectories eventually go to?
What kind of attractor is it?

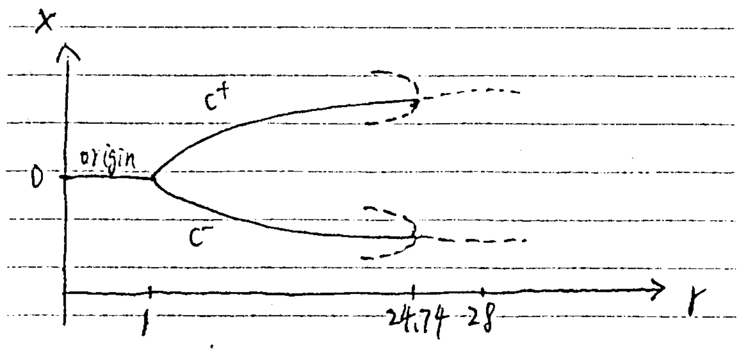
Next, we are going to follow Lorenz's steps to the big discovery of chaos and strange attractors.

Chaos on a Strange Attractor.

Lorenz numerically integrated those equations first with the choice of parameters as

$$\sigma = 10, \quad b = \tfrac{8}{3}, \quad r = 28.$$

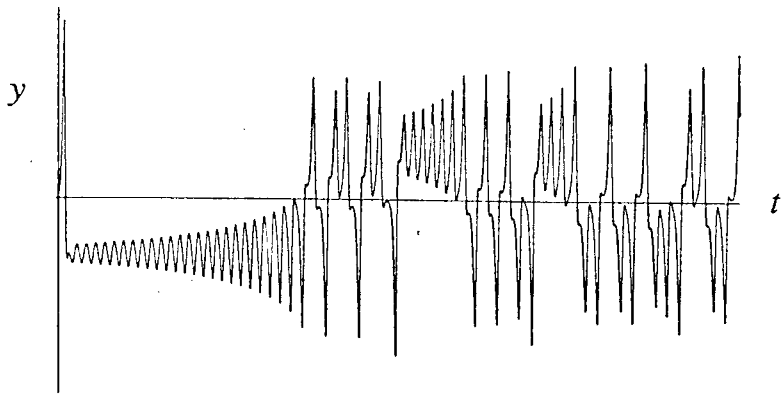
Note that when $\sigma = 10$, $b = \frac{8}{3}$, $r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1} \approx 24.74.$



So his choice of $r = 28$ is just above $r_H \approx 24.74$ and clearly this is the case we are interested in (he was interested in it first).

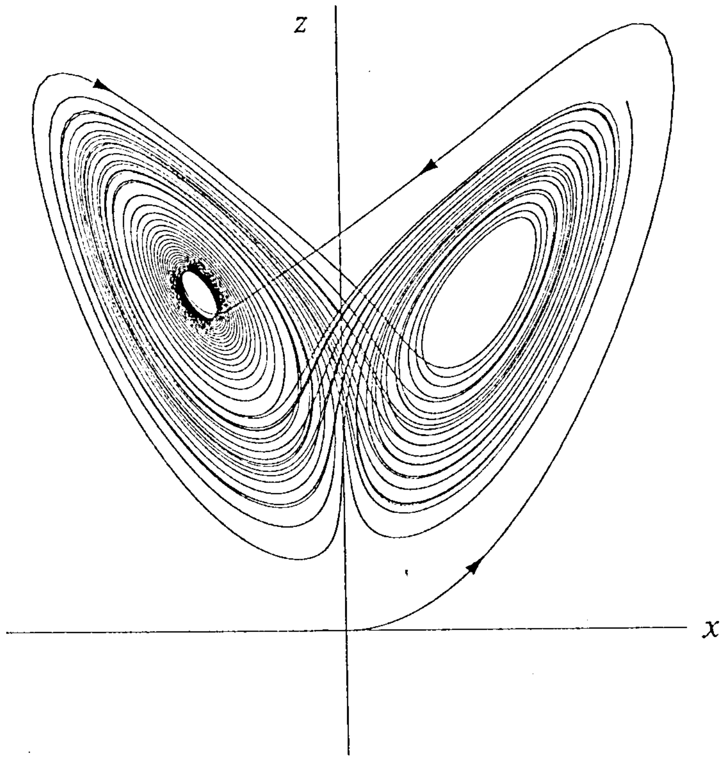
This is what he found:

1. After an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly. The motion is aperiodic.



2. The solution is very sensitive to the initial conditions. A very small change in initial conditions can result in a big change in the solutions.

3. When he plotted the solution in the (x, y, z) phase space, a beautiful butterfly structure emerges (see the separate page). This is the famous Lorenze attractor. Its projection on the (x, z) phase is as illustrated on the next page.



Observation: The trajectory starts near the origin, then swings to the right, and then dives into the center of a spiral on the left. After a very slow spiral outward, the trajectory shoots back over to the right side, spirals around a few times, shoots over to the left, spirals around, and so on indefinitely. The number of circuits made on either side varies unpredictably from one cycle to the next.

Lorenz Attractor