Applications of Regular Perturbation to Airbags

Alex Beals, Jeff Kim, Aaron Lewin, Alma Wang | MATH 046 | Spring 2016 | Advisor: Professor Nishant Malik

Abstract

In this project, we will analyze the chemistry and physics behind airbags during head on vehicle collisions. This project is the culminating experience for Math 46 (Introduction to Applied Math), taught by Professor Nishant Malik. We first apply perturbation theory and methods of solving differential equations to the two-part airbag chemical system. Then we analyze the car-wall and human-airbag physical systems assuming the wall and airbag act as damping agents.

The project uses differential reaction rate equations to determine the concentration of compounds at a given time. By rearranging differential equations, we are able to set the value of each compound to a linear combination of NaN_3, K_2O and initial conditions K_0 and N_0. We apply leading-order approximation perturbation methods to derive solutions to NaN_3 and K_2O. These solutions are compared to values approximated using Euler's method.

We then predict the damping effectiveness of the airbags from a physics perspective by comparing time that the driver decelerates versus the time the car decelerates. We assume the airbag acts as a damping agent with increased damping ability over time. By making assumptions about the behavior of the damping, we construct an equation for damping of the airbags.

Background

The airbag was introduced as a means of displacing force during a collision in order to minimize injury through rapid inflation of a compressed device. When a vehicle undergoes a head-on collision, an accelerometer produces an electrical impulse that heats sodium azide (NaN_3) to 200°C. The decomposition sodium azide produce pure sodium metal (Na) and nitrogen gas (N_2). The nitrogen gas inflates the airbag. As a safety measure, a secondary reaction combines highly reactive pure sodium with potassium nitrate (KNO_3) to create solids that are transformed into alkaline silicate glass which is stable and unreactive (Casiday, 2016). We analyze the first two of three equations in the airbag chemical system:

$$2NaN_3 \xrightarrow{k_1} 2Na + 3N_2$$
$$10Na + 2KNO_3 \xrightarrow{k_2} K_2O + 5Na_2O + N_2$$

Figure 1

The sodium azide reaction requires a high activation energy to decompose, so without additional heat, it remains stable when the car is operating at normal conditions.

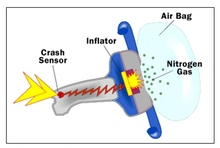

Figure 2.a: Diagram of airbag Mechanism

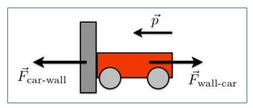

Figure 2.b: Forces during a car collision
http://www.wired.com/wp-content/uploads/2010/05/i-d2c31183355d14261adbe08756bbd5388-2010-05-06_untitled_4.jpg

The law of mass action states the rate of an elementary chemical equation is proportional to the concentration of reactants (Howard, 2009). Therefore the reaction rate equals a constant k_i multiplied by the concentrations of each reactant raised to the power of its coefficient. Since the law of mass action determines rate per unit of total product, the rate of change for a particular compound depends on the stoichiometry of the chemical equation.

In part two of the project, we study the effect of damping on the car and the driver in a head on collision. Under this scenario, we investigate the motion of the car and the passenger by assuming a different damping term for each subject: the wall damps the motion of the car, and the airbag damps the motion of the driver.

$$M_{car}\frac{dv_1}{dt} + b_1 v_1 = 0, v_1(0) = 30mph = 13.4\frac{m}{s}$$
$$M_{person}\frac{dv_2}{dt} + b_1(1 - e^{-at})v_2 = 0, v_2(0) = 30mph = 13.4\frac{m}{s}$$

Figure 3

For simplicity, we assume the driver is not wearing a seatbelt so the airbag is the only force acting to decelerate the driver in the collision. When the car collides with the wall the driver continues to move at the initial velocity of the car until he makes contact with the fully inflated airbag as evident from the delay in the decay of the driver's velocity. The airbag deflates as the driver slows to a stop. The model shows the damping effect on the car as a constant value while the value of the damping effect on the driver varies with time due to the airbag to decrease the force on the driver as during the deceleration phase.

Our Research

Deriving Concentration Approximations

Following from Fig. 1, we assume that the reactions are irreversible, due to their high activation energy, and are elementary equations, meaning the rate of change of the compound quantities can be written in the following way:

$$\frac{d[Na]}{dt} = 2k_1[NaN_3]^2 - 10k_2[Na]^{10}[KNO_3]^2 \qquad [Na](0) = 0$$
$$\frac{d[NaN_3]}{dt} = -2k_1[NaN_3]^2 \qquad [NaN_3](0) = N_0$$
$$\frac{d[KNO_3]}{dt} = -2k_2[Na]^{10}[KNO_3]^2 \qquad [KNO_3](0) = K_0$$
$$\frac{d[Na_2O]}{dt} = 5k_2[Na]^{10}[KNO_3]^2 \qquad [Na_2O](0) = 0$$
$$\frac{d[N_2]}{dt} = 3k_1[NaN_3]^2 + k_2[Na]^{10}[KNO_3]^2 \qquad [N_2](0) = 0$$
$$\frac{d[K_2O]}{dt} = k_2[Na]^{10}[KNO_3]^2 \qquad [K_2O](0) = 0$$

Figure 4

Using the principle of conservation of mass, the equations can be reworked into relationships for all six compounds in terms of only two compounds, NaN_3 and K_2O:

$$Na = N_0 - NaN_3 - 10K_2O$$
$$NaN_3 = NaN_3$$
$$KNO_3 = K_0 - 2K_2O$$
$$Na_2O = 5K_2O$$
$$N_2 = -\frac{3}{2}NaN_3 + K_2O$$
$$K_2O = K_2O$$

Figure 5

The system reduces to two first-order differential equations. We can solve analytically for NaN_3 in Fig. 6.a. K_2O follows a nonlinear equation. We approximate using leading-order perturbation resulting in Fig. 6.b:

$$[NaN_3] = \frac{1}{2k_1 t + N_0^{-1}}$$

Figure 6.a

$$[K_2O] = \frac{N_0 - [NaN_3]}{10}$$

Figure 6.b

Fig. 7 below displays approximate concentrations over the first second using arbitrary rate constants ($k_{1,2}$= 1) and initial concentrations ([NaN_3] = five mol, [K_2O] = eight mol). Fig. 7.a approximates using Euler's approximation. Fig. 7.b display our approximate derivations shown in Fig. 5 above.

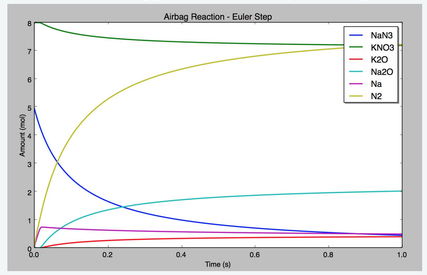

Figure 7.a

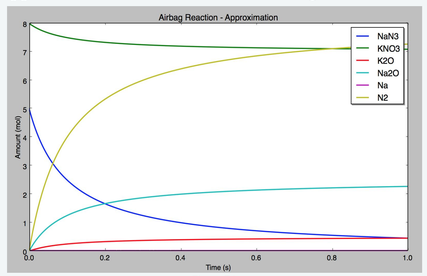

Figure 7.b

Estimating Rate Constants

We estimate the rate constants using graphical analysis. We determine the N_2 concentration necessary (using PV = nRT) to fill an airbag to 70L (the appropriate volume to protect the driver) to be 1.8 mol at 200°C. According to Merola (1999), 130g NaN_3 (2.00 mol) is stored in the car prior to the airbag deploying. We assume a large value for initial K_2O concentration to ensure it would be non-limiting. According to Huffman (2011), the time between impact and the time that an airbag must reach full capacity is 23ms, so the reactions must create 70L of N_2 gas in 23 ms. Because most of the concentration of N_2 is created by the first reaction, we assumed values of k_2 to solve for k_1. At t = 23ms, [N_2] must be 1.8 mol. Using the slider function in m46_euler.py, we show that if we vary k_2 between 10 and 10000, k1 varies only from 14.53 to 13.99. We assume that k_2 = 10 and k_1 = 14.53 to ensure the driver's safety. We believe that k_2 = 10000 is unlikely.

Airbags as a Damping Agent

After deriving the differential equations for the chemical system inside an airbag, we examined the damping effects of the airbag on the driver. We modeled the both the car and the driver decelerating as two systems starting with the same initial velocity. As discussed in the background section, the damping coefficient of the driver is not constant. The maximum damping coefficient acting on the driver would be as if there was no airbag. We treated this maximum as the constant damping coefficient on the car. We assume that the damping coefficient is initially zero, and approaches this maximum. The analytical solutions are:

$$v_1(t) = v_1(0)e^{\frac{-b_1(t)}{M_{car}}} \qquad v_2(t) = \begin{cases} v_2(0) & \text{if } t < 23\text{ms} \\ v_2(0)e^{\frac{b_1}{aM_{person}}(1-e^{-a(t-23ms)}-a(t-23ms))}, & \text{otherwise} \end{cases}$$

Figure 8

Using graphical analysis and kinematic equation in Fig 9:

$$\Delta X = v_i t + \frac{1}{2}at^2, a = \frac{v_f - v_i}{t}$$

Figure 9

We determined that for the car (1000 kg) to stop one meter after the collision, the car must come to a stop after 149ms. Therefore, we matched the constant damping coefficient to be b_1 = 40000 kg/s.

The constant b_1 is used in the v_2 equation, with 'a' found to be 0.09 through graphical analysis for a given driver mass (70kg). This results in a stopping time for the driver of approximately 0.5 seconds, a safer deceleration than with no airbag.

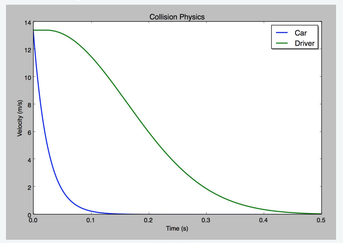

Figure 10

Conclusions and Future Work

Our perturbed approximations yield results semi-independent of k_2. The error is small when comparing the Euler approximation to our perturbation approximations. Interestingly, our approximation yields [Na] = 0 for all time. This supports our intuition that the highly reactive Na is quickly depleted, showing the system to be safe.

Our project relies heavily on estimates for rates constants and damping coefficients based on outside sources or graphical analysis. We believe our approximations could be significantly more accurate if we had the option to do trial experiments to determine the actual rate constants of the chemical equations and actual damping coefficients of the physical systems. We could explore the use of a catalyst to increase the reaction rate, causing the time between when the crash and full airbag deployment to be lower, therefore safer for the driver in a head on collision. The estimates for our differential equations rely on the chemical reactions as elementary reactions. Verifying this chemisty property provides additional support for our work.

References

(1) Howard, P. *Modeling with ODE* (Fall 2009): n. pag. Texas A&M University. Web. (2) Woodward, Chris. "How Do Airbags Work?" *Explain That Stuff*. N.p., 23 Sept. 2015. Web. 29 May 2016. (3) Casiday, Rachel. "Gas Laws Save Lives: The Chemistry Behind Airbags."*Chemistry Behind Airbags*. Washington University, Web. 29 May 2016. (4) Huffman, John Purley. "The Physics Of: Airbags - Feature." - *Car and Driver*. Car and Driver, June 2011. Web. 29 May 2016. (5) Brain, Marshall. "How Airbags Work." *HowStuffWorks*. N.p., 01 Apr. 2000. Web. 29 May 2016. (6) Howard, P. "Modeling with ODE." *Elementary Reaction* (2009): n. pag. Print. (7) How Airbags Work." *HowStuffWorks*. N.p., 01 Apr. 2000. Web. 30 May 2016. (8) Merola, Joseph. *Scientific American*. Nature America Inc., 1999. Web. (8) Casiday, Rachel. "The Chemistry in Airbags." *The Chemistry in Airbags*(2009): n. pag. Web. May 2016.