Worksheet #7: Dominant balancing

(1) Find the scaling of x with ϵ that makes two terms of equal order and others of lower order, in:

$$\epsilon x^4 + \epsilon x^3 - x^2 + 2x - 1 = 0$$

Guess1: $\varepsilon x^4 \sim x^2 \rightarrow x = O(1/\sqrt{\varepsilon}) \Rightarrow \varepsilon x^3 \sim O(1/\sqrt{\varepsilon})$ $2x \sim O(1/\sqrt{\varepsilon})$ $-1 \leftarrow O(1)$

Guess2: $\varepsilon x^4 \sim 2x \rightarrow x = O(\varepsilon^{-1/3}) \rightarrow \varepsilon x^3 = O(1)$ $x^2 = O(\varepsilon^{-2/3})$

Guess 3: $\varepsilon x^4 \sim 1 \rightarrow x = O(\varepsilon^{-1/4}) \rightarrow x^2 \rightarrow O(\varepsilon^{-1/2})$ $2x = O(\varepsilon^{-1/4})$

Guess 1 is the best option. everything scales about the same.

(2) Find the leading order term in each of the four roots.

let $y = \sqrt{\varepsilon} x \rightarrow x = y/\sqrt{\varepsilon} \rightarrow$ Polynomial becomes.

$$\frac{\varepsilon y^4}{\varepsilon^2} + \frac{\varepsilon y^3}{\varepsilon^{3/2}} - \frac{y^2}{\varepsilon} + \frac{2y}{\sqrt{\varepsilon}} - 1 = 0 \Rightarrow y^4 + \sqrt{\varepsilon} y^3 - y^2 + 2\sqrt{\varepsilon} y - \varepsilon = 0.$$

leading order behavior is found by setting $\varepsilon = 0$.

$\Rightarrow y^4 - y^2 = 0 \rightarrow y = 0, \pm 1$ Toss $y=0$ since it is clearly not an approximate root.

to obtain higher order terms look for series solutions s.t.

$$y = y_0 + \varepsilon^{1/2} y_1 + \varepsilon y_2 + \varepsilon^{3/2} y_3 + \cdots$$

Plug in and solve for $y_0, y_1,$ etc.