Worksheet #18: Green's identities

(1) Construct the product rule for the div operator. In other words, what is $\nabla \cdot (u\bar{J})$? [Hint: look for ∇u]

$$\nabla \cdot (uJ) = \left(\frac{d}{dx_1}, \frac{d}{dx_2}\right) \cdot (uJ_1, uJ_2)$$

$$= \frac{d}{dx_1}(uJ_1) + \frac{d}{dx_2}(uJ_2) = \underline{\frac{du}{dx_1} J_1} + \frac{dJ_1}{dx_1} u + \underline{\frac{du}{dx_2} J_2} + \frac{dJ_2}{dx_2} u$$

$$= J \cdot (\nabla u) + (\nabla \cdot J) u$$

(2) Write out $\nabla \cdot (u\bar{J})$ for $\bar{J} = \nabla v$.

$$\nabla \cdot (u \nabla v) = \nabla v \cdot \nabla u + (\nabla \cdot \nabla v) u = \nabla v \cdot \nabla u + u \Delta v$$

(3) Integrate this expression over Ω and use the Divergence theorem. You should get Green's first identity.

$$\int_\Omega (\nabla v \cdot \nabla u + u \Delta v) dx = \int_\Omega \nabla \cdot (u \nabla v) dx$$

$$= \int_{\partial\Omega} \bar{n} \cdot (u \nabla v) ds = \int_{\partial\Omega} u (\bar{n} \cdot \nabla v) ds$$

$$= \int_{\partial\Omega} u \frac{dv}{dn} ds$$

(4) From this identity, subtract the same identity with u and v swapped. This leads you to Green's second identity.

$$\int_\Omega (\nabla v \cdot \nabla u + u \Delta v) dx = \int_{\partial\Omega} u \frac{dv}{dn} ds$$

$$- \quad \int_\Omega (\nabla u \cdot \nabla v + v \Delta u) dx = - \int_{\partial\Omega} v \frac{du}{dn} ds$$

$$\int_\Omega (u \Delta v - v \Delta u) dx = \int_{\partial\Omega} \left(u \frac{dv}{dn} - v \frac{du}{dn}\right) ds$$