Worksheet #15: Image deblurring (1D)

Consider the symmetric blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y)f(y)dy$, where $k(s)$ is even, symmetric, and 2π-periodic. $k(s)$ is called an aperture function.

(1) Show that $\phi_n(x) = 1$ is an eigenfunction of K, and find its eigenvalue? [Hint: why is $K\phi_n(x)$ independent of x? Why is λ_0 independent of x?]

$$(K1)(x) = \int_{-\pi}^{\pi} k(x-y)\,1\,dy = \int_{-\pi-x}^{\pi-x} k(-s)ds = \int_{-\pi}^{\pi} k(s)ds.$$

$s = y-x$

$$\Rightarrow \lambda_0 = \text{const} := \int_{-\pi}^{\pi} k(s)ds$$

(2) Show that $\phi_n(x) = \cos(nx)$, $n = 1, 2, \ldots$ is an eigenfunction of K, find its eigenvalue of λ_n. [Hint: use addition formula, k even]

$$(K\phi_n)(x) = \int_{-\pi}^{\pi} k(x-y)\cos(ny)dy = \int_{-\pi-x}^{\pi-x} k(-s)\cos(n(s+x))ds$$

$$= \int_{-\pi}^{\pi} k(s)\cos(ns)\cos(nx) - \int_{-\pi}^{\pi} k(s)\sin(ns)\sin(nx)ds$$

$$= \cos(nx)\int_{-\pi}^{\pi} k(s)\cos(ns)ds = \lambda_n \phi_n(x).$$

(3) How do λ_n relate to Fourier cos coefficients K_n of aperture function $k(s)$?

$$\lambda_n = \int_{-\pi}^{\pi} k(s)\cos(ns)\,ds = \pi \underbrace{k_n}_{\text{fourier coeifficients of } k(s)}$$

You could check that $\sin(nx)$ is also eigenfunction with same eigenvalue λ_n. Assume image is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ and $(Kf)(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n\cos(nx) + B_n\sin(nx)]$

(4) How are g's Fourier coefficients related to those of f?

Note the Fourier basis is the eigenbasis for K. so the action of K is multiplication on its fourier coeffients.

$\rightarrow A_0 = \lambda_0 a_0 = \pi k_0 a_0$ $\qquad B_n = \lambda_n b_n = \pi k_n b_n$

$A_n = \lambda_n a_n = \pi k_n a_n$

Such is the nature of convolution kernels. How would you invert $g \to f$ ie. deconvolve?

Recall $g = Kf$ so if we want to recove the image given g. we must divide each of the fourier coeifficents by πk_n.

ie $a_n = \dfrac{A_n}{\pi k_n}$ $\qquad b_n = \dfrac{B_n}{\pi k_n}$

This is called deconvolving