Worksheet #13: Volterra integral equations

(1) Convert the following integral equation into an IVP for $u(t)$.

$$(*) \quad \int_0^t yu(y)dy - \alpha u(t) = f(t) \qquad \text{on } 0 \leq t \leq 1$$

by Fundamental Thm of calc $\frac{d}{dt}\left(\int_0^t y\,u(y)dy\right) = t\,u(t)$

Now we can differentiate the equation, to find

$$t\,u(t) - \alpha u'(t) = f'(t) \rightarrow u'(t) - \frac{t}{\alpha}(u(t)) = \frac{-f'(t)}{\alpha}$$

to get I.C plug $t=0$ into $(*)$ $\Rightarrow -\alpha u(0) = f(0) \rightarrow u(0) = \frac{f(0)}{-\alpha}$

Can you solve the IVP? Yes. the soln is

$$u(t) = -\alpha\, e^{t^2/2\alpha}\left(\int_0^t e^{-s^2/2\alpha} f'(s)ds + f(0)\right)$$

(2) Prove the lemma:

$$\int_a^x \int_a^s f(y)dyds = \int_a^x f(y)(x-y)dy$$

[Hint: Let $F(s) = \int_a^s f(y)dy$ and use integration by parts.]

Rewrite the integral: $\int_a^x F(s)ds$ let $u = F(s)$, $du = F'(s)ds$, $v = s$, $dv = 1\,ds$

$$\rightarrow \int_a^x \left(\int_a^s f(y)dy\right)ds = \int_a^x F(s)ds$$

$$= sF(s)\Big|_a^x - \int_a^x sF'(s)ds$$

$$= xF(x) - aF(a) - \int_a^x sF'(s)ds.$$

$$= x\int_a^x f(s)ds - (0) - \int_a^x s\,f(s)\,ds$$

$$= \int_a^x (x-s)\,f(s)\,ds.$$

(3) Convert the IVP

$$\begin{cases} u''(t) + q(t)u(t) &= g(t) \\ u(0) &= A \\ u'(0) &= B \end{cases}$$

into a Volterra integral equation of the form $Ku - \lambda u = f$ where Ku is an integral operator.

Integrating twice we get

$$u(t) - A - Bt + \int_0^t \int_0^s q(y)u(y)\,dy\,ds = \int_0^t \int_0^s g(y)\,dy\,ds$$

using part 2 we can rewrite as

$$u(t) - A - Bt + \int_0^t (t-s)q(s)u(s)\,ds = \int_0^t (t-s)\,g(s)\,ds$$

$\lambda = -1$

$f(t) = \int_0^t (t-s)g(s)ds + Bt + A$

$k(t,s) = (t-s)\,q(s)$

$\Rightarrow (Ku)(t) = \int_0^t k(t,s)u(s)ds$

(4) Now try to convert

$$u''(t) + p(t)u'(t) + q(t)u(t) = g(t)$$

into a second kind Volterra integral equation.

Integrate

$$u'(t) - u'(a) = -\int_a^t p(s)\underbrace{u'(s)ds}_{dv} - \int_a^t (q(s)u(s) + g(s))ds$$

$$= -\left[p(s)u(s)\Big|_a^t - \int_a^t u(s)p'(s)\,ds\right] - \int_a^t (q(s)u(s) + g(s))ds$$

$$= -p(t)u(t) + p(a)u(a) + \int_a^t (p'(s) - q(s))u(s)ds$$

Integrate again

$$u(t) - u(a) - u'(a)(t-a) = -\int_a^t p(s)u(s)ds + p(a)u(a)(t-a) + \int_a^t \left(\int_a^s (p'(y) - q(y))u(y)dy\right)ds$$

Finally $u(t) = u(a) + (u'(a) + p(a)u(a))(t-a) - \int_a^t p(s)u(s)ds$

$$+ \int_a^t (p'(s) - q(s)) u(s)\,(t-s)\,ds$$