Worksheet #12: Real or not... The story of Sturm-Liouville eigenvalues

Consider the Sturm-Liouville problem with $p = 1$ and $q(x)$ real:

$$-y'' + q(x)y = \lambda y, \qquad a < x < b$$

with Dirichlet boundary conditions $y(a) = y(b) = 0$.

(a) Multiply the ODE by $\bar{y}$.

$$-\bar{y}\, y'' + q(x)\, y\bar{y} = \lambda y \bar{y}$$

(b) Multiply the conjugate of the ODE by y.

$$-y\,\bar{y}'' + q(x)\,\bar{y}\, y = \bar{\lambda} y \bar{y}$$

(c) Subtract the two equations. (There should be some cancellation.)

$$-(\bar{y} y'' - y \bar{y}'') = (\lambda - \bar{\lambda})\, y\bar{y}$$

(d) Integrate over the interval (a, b). [Hint: use integration by parts]

$u = \bar{y}$ $\quad dv = y''dx$

$du = \bar{y}'dx$ $\quad v = y'$

Integrate by parts

$$-\int_a^b (\bar{y} y'' - y\bar{y}'')dx = (\lambda - \bar{\lambda})\int_a^b y\bar{y}\, dx$$

$$-\left[\bar{y} y' - y\bar{y}'\right]\Big|_a^b - \int_a^b \underbrace{(\bar{y}'y' - y'\bar{y}')}_{0} dy = 0$$

(e) Apply boundary conditions.

$-[\bar{y}y' - y\bar{y}']\big|_a^b = 0$ by BC.

$\Rightarrow (\lambda - \bar{\lambda})\int_a^b y\bar{y}\, dx = 0$

(f) What is the sign of $\int_a^b y\bar{y}dx$? [Hint: If a is a complex number (ie. $a = b + ci$ for b and c real constants) then $a\bar{a} = (b + ci)(b - ci)$]

let $y = f(x) + i g(x)$. then $y\bar{y} = (f(x))^2 + (g(x))^2 \geq 0 \;\; \forall x$

$\Rightarrow \int_a^b \bar{y}y\, dx \geq 0 \;\; \forall x.$ only way to $= 0$ if $f = g = 0$.

(g) Conclude something about $\lambda - \bar{\lambda}$. What does this mean about λ?

Therefor $\lambda - \bar{\lambda}$ must be zero.

$\lambda = \alpha + \beta i \qquad \lambda - \bar{\lambda} = \alpha + \beta i - (\alpha - \beta i)$

$= 2\beta i = 0 \Rightarrow \beta = 0.$

$\Rightarrow \lambda$ is a real number.

(h) What other boundary conditions would this work for? Neumann? Periodic? ($y(a) = y(b)$ and $y'(a) = y'(b)$) Mixed ($y'(a) = \alpha y(a)$ and $y'(b) = \beta y(b)$)

This will be true for ~~N~~ Neumann BC.

Check periodic.

$$\bar{y}y' - y\bar{y}'\Big|_a^b = \bar{y}(b)y'(b) - y(b)\bar{y}'(b) - \left(\bar{y}(a)y'(a) - \bar{y}'(a)y(a)\right)$$

$$= 0 \checkmark$$

Therefor true.

Check the boundary term for mixed BC.

$$\bar{y}y' - y\bar{y}'\Big|_a^b = \bar{y}(b)y'(b) - y(b)\bar{y}'(b) - \left(\bar{y}(a)y'(a) - y(a)\bar{y}'(a)\right)$$

$$= \bar{y}(b)\beta y(b) - y(b)\bar{\beta}\bar{y}(b) - \left(\bar{y}(a)\alpha y(a) - y(a)\bar{\alpha}\bar{y}(a)\right)$$

$$= 0 \text{ if } \alpha \,\&\, \beta \text{ are real.}$$