Worksheet #10: WKB eigenvalues

Consider the boundary value problem

$$-y'' = \lambda q(x) y \qquad \text{where } y(0) = y(1) = 0.$$

(1) Transform the equation into the form

$$\epsilon^2 y'' + (k(x))^2 y = 0.$$

What are ϵ and $k(x)$?

$\varepsilon = 1/\sqrt{\lambda} \qquad k(x) = \sqrt{q(x)}$

(2) Will WKB apply for small or large λ?

WKB works for small $\varepsilon \Rightarrow$ Large λ.

(3) Use WKB to give an approximation of the n^{th} eigenvalue λ_n for the problem

$$-\frac{1}{(2-x^2)^2} y'' = \lambda y \qquad \text{where } y(0) = y(1) = 0.$$

$\Rightarrow \varepsilon^2 y'' + (2-x^2)^2 y = 0$

$$\lambda_n = \left[n\pi \left(\int_0^1 k(x)\,dx\right)^{-1}\right]^2 = \left[n\pi \left(\int_0^1 (2-x^2)\,dx\right)^{-1}\right]^2 = \left[n\pi\left(2x - \frac{x^3}{3}\Big|_0^1\right)^{-1}\right]^2$$

$$\cong \left(\frac{3\pi n}{5}\right)^2$$

(4) What are the WKB eigenfunctions?

$$y_n(x) = \frac{1}{\sqrt{k(x)}} \sin\left(\sqrt{\lambda_n} \int_0^x k(s)\,ds\right)$$

$$= \frac{1}{\sqrt{2-x^2}} \sin\left(\frac{3\pi}{5} n \int_0^x (2-s^2)\,ds\right)$$

$$= \frac{1}{\sqrt{2-x^2}} \sin\left(\frac{3n\pi\,(2x - x^3/3)}{5}\right)$$