Math 46 Spring 2013

Introduction to Applied Mathematics

Second Midterm Exam

Thursday, May 16, 5:00-7:00 PM

Your name (please print): ____________________________________

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

____

For grader use only:

Problem	Points	Score
1	6	
2	10	
3	6	
4	8	
5	6	
6	6	
7	8	
Total	50	

1. [6 points] Find the first two terms in the asymptotic expansion of $I(x) = \int_x^\infty e^{-t^4} dt$ for x large ($x \to \infty$). [Hint: $e^{-t^4} = \frac{1}{t^3}\frac{d}{dt}\left(e^{-t^4}\right)$]

[BONUS: prove that the remainder term satisfies the needed condition for an asymptotic expansion]

2. [10 points]

(a) What are the eigenvalues and eigenfunctions for the integral operator

$$[Ku](x) = \int_0^1 xy^3 u(y)dy?$$

(b) Solve the integral equation $[Ku](x) - u(x) = x^4$ on $(0,1)$, or explain why it is not possible.

(c) Solve the integral equation $[Ku](x) = x$ on $(0,1)$, or explain why it is not possible.

(d) Solve the integral equation $[Ku](x) = x^2$ on $(0,1)$, or explain why it is not possible.

3. [6 points] Consider the integral operator $[Ku](x) = \int_2^{2e} k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} 1 - \ln\left(\frac{y}{2}\right), & x < y \\ 1 - \ln\left(\frac{x}{2}\right), & y < x \end{cases}$$

Convert the eigenvalue problem $Ku = \lambda u$ into a Sturm-Liouville problem on the interval $(2, 2e)$. Do not forget to find homogeneous boundary conditions. [Hint: one will be Dirichlet, one Neumann]

4. [8 points] Consider the boundary-value problem $-u''(x) + \omega^2 u(x) = f(x)$ for $\omega^2 > 0$ (a fixed constant) on the interval $x \in [0, 1]$ with Dirichlet boundary conditions $u(0) = u(1) = 0$.

   (a) Can a Greens function exist for this problem? (Why?)

   (b) If the Greens function can exist, find it. Otherwise solve the problem for general $f(x)$ another way.

5. [6 points] Find the first 2 non-zero terms in the Neumann series solution of the following Volterra integral equation.

$$u(x) = e^x + \int_0^x e^{y-x} u(y)dy$$

6. [6 points] What can be deduced about the sign of the eigenvalues of

$$y'' + x^3 y = \lambda y$$

with boundary conditions $y(-1) = y(0) = 0$?

7. [8 points]

(a) Determine if there is a Green's function associated with the operator $Lu = u''+9u$, $0 < x < \pi$, with $u(0) = u(\pi) = 0$.

(b) Assuming $f(x) \in L^2([0, \pi])$, find all solutions to the boundary value problem

$$u'' + 9u = f(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

Useful formulae

non-oscillatory WKB approximation

$$y(x) = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\epsilon} \int k(x) dx}$$

Binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

‘ Addition formulae

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta
\end{aligned}$$

Leibniz’s rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)\,dy\right) = \left(\int_{a(x)}^{b(x)} f_y(x,y)\,dy\right) + f(x,b(x))b'(x) - f(x,a(x))a'(x)$$