Math 46 Spring 2013

Introduction to Applied Mathematics

First Midterm Exam

Wednesday, April 24 or Thursday, April 25, 5:00-7:00 PM

Your name (please print): Solutions

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

For grader use only:

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. The length L of an organism depends upon time t, its density ρ, its resource assimilation rate a (mass per area per time), and its resource use rate b (mass per volume per time). Show that there is a physical law involving two dimensionless quantities.

Fundamental units.

$[L] = \ell$ $\quad [\rho] = M\ell^{-3}$ $\quad [b] = ML^{-3}T^{-1}$

$[t] = T$ $\quad [a] = M\ell^{-2}T^{-1}$

Dimension matrix

$$\begin{array}{c} \\ M \\ \ell \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} L & t & \rho & b & a \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & -3 & -3 & -2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Size of $A = 3 \times 5$

rank of $A = 3$.

$\Rightarrow$ # of dimensionless quantities $= 5 - 3 = 2$.

2. The temperature $\Theta = \Theta(t)$ of a c chemical sample in a furnace at time t is governed by the initial value problem

$$\frac{d\Theta}{dt} = qe^{-A/\Theta} - k(\Theta - \Theta_f), \qquad \Theta(0) = \Theta_0$$

where Θ_0 is the initial temperature of the sample, Θ_f is the temperature of the furnace, and q, k, and A are positive constants. The first term on the right hand side is the heat generation term, and the second is the heat loss term given by Newton's law of cooling.

(a) What are the dimensions of the constants q, k, and A? [Hint: make the exponential dimensionless.]

$[q] = \frac{\Theta}{T}$ $\quad [k] = \frac{1}{T}$ $\quad [A] = \Theta$.

(b) What are possible time scales? want t/t_c unitless

$t_c = k^{-1}$ $\qquad t_c = \frac{A}{q}$

(c) Reduce the problem to dimensionless form using Θ_f as the temperature scale. Choose the time scale to be appropriate for the case when the heat loss term is large compared to the heat generated term. Define an ϵ to be a small dimensionless parameter.

Let $W = \frac{\Theta}{\Theta_F}$ $\quad \tau = t/t_c$. Plug into equation

$$\frac{\Theta_F W'}{t_c} = qe^{-A/(\Theta_F W)} - k(\Theta_F W - \Theta_F)$$

$$W' = \frac{t_c q}{\Theta_F} e^{-A/(\Theta_F W)} - t_c k(W-1)$$

we want the 2nd term larger than the 1st.

$\Rightarrow$ take $t_c = 1/k$. $\Rightarrow$ $\boxed{\varepsilon = \frac{q}{\Theta_F k}}$ $\quad$ let $\alpha = A/\Theta_F$

$$W' = \varepsilon e^{-\alpha/W} - (W-1) \qquad W(0) = \Theta_0/\Theta_F$$

3. Consider the equation of motion for a conservative oscillator with a cubic restoring force (the Duffing equation)

$$u'' + 9u = 3\epsilon u^3$$

with initial conditions $u(0) = a > 0$ and $u'(0) = 1$. ϵ is small. Find a two term (non-growing) approximation of the solution.

We need to use Poincare-Lindstedt

let $y(\tau) = u(\omega t)$ $\quad \tau = \omega t$ where $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2$

Plug into the problem

$\omega^2 y'' + 9y = 3\varepsilon y^3 \qquad y(0) = a \quad y'(0) = 0$

Use regular perturbation now. $y = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$

$(1+\omega_1\varepsilon+\cdots)^2[y_0'' + \varepsilon y_1'' + \cdots] + 9(y_0 + \varepsilon y_1 + \cdots) = 3\varepsilon(y_0 + \varepsilon y_1 + \cdots)^3$

Collect equations

ε^0: $y_0'' + 9y_0 = 0 \qquad$ BC $y_0(0) = a \quad y'(0) = 0$

$y_0(\tau) = a_1\cos(3\tau) + a_2\sin(3\tau) \qquad$ By BC. $a_1 = a$, $a_2 = 0$.

$\to y_0(\tau) = a\cos(3\tau)$

ε^1: $2\omega_1 y_0'' + y_1'' + 9y_1 = 3y_0^3 \qquad$ BC $y_1(0) = 0 \quad y'(0) = 0$.

$y_1'' + 9y_1 = 3a^3\cos^3(3\tau) + 18a\omega_1\cos(3\tau)$

$= \frac{3}{4}a^3(3\cos(3\tau) + \cos(9\tau)) + a18\omega_1\cos(3\tau)$

Choose ω_1 st $\frac{9}{4}a^3 + a18\omega_1 = 0$

$\to \omega_1 = -\frac{9a^2}{4(18)} = -\frac{a^2}{8}$

So ε^1 equation becomes

$y_1'' + 9y_1 = \frac{3a^3}{4}\cos(9\tau)$

homogeneous: $y_1^h = C_1\cos(3\tau) + C_2\sin(3\tau)$

Particular: $y_1^p = A\cos 9\tau + B\sin(9\tau)$

Plugin to find. $B = 0 \quad A = -\frac{a^3}{96}$

Now enforce Boundary conditions to find

$C_2 = 0$

$C_1 = \frac{a^3}{96}$

$\Rightarrow u(t) = a\cos(3\omega t) + \frac{\varepsilon a^3}{96}\left(\cos(3\omega t) - \cos(9\omega t)\right)$

4. Find the WKB approximation to the problem

$$\epsilon^2 y'' - (2+x)^2 y = 0, \qquad x \geq 0 \qquad k(x) = 2+x$$

such that $y(0) = 1$ and $\lim_{x\to\infty} y(x) = 0$.

$$y_{WKB}(x) = \frac{C_1}{\sqrt{2+x}} e^{\frac{1}{\varepsilon}\int 2+x\, dx} + \frac{C_2}{\sqrt{2+x}} e^{-\frac{1}{\varepsilon}\int 2+x\, dx}$$

$$= \frac{C_1}{\sqrt{2+x}} e^{\frac{1}{\varepsilon}(2x+x^2/2)} + \frac{C_2}{\sqrt{2+x}} e^{-1/\varepsilon\,(2x+x^2/2)}$$

$$y(0) = \frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} = 1 \rightarrow C_2 = \sqrt{2}$$

$$\lim_{x\to\infty} y(x) = 0 \rightarrow C_1 = 0$$

$$y_{WKB}(x) = \sqrt{\frac{2}{2+x}}\, e^{-(2x+x^2/2)/\varepsilon}$$

5. Short answer

(a) Does $\epsilon\cos(\epsilon t)$ converge uniformly to 0 as $\epsilon \to 0^+$ for $t \in [0,\infty)$? Briefly explain your answer.

$\max\limits_{t \in [0,\infty)} |\varepsilon \cos(\varepsilon t)| = \varepsilon \qquad \lim\limits_{\varepsilon \to 0^+} \varepsilon = 0$

$\to$ It converges uniformly.

(b) Show that $\ln \epsilon = o(\epsilon^{-p})$ as $\epsilon \to 0^+$ for all $p > 0$.

$$\lim_{\varepsilon \to 0^+} \frac{\ln \varepsilon}{\varepsilon^{-p}} \overset{LH}{=} \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-p \cdot \varepsilon^{-p-1}} = \lim_{\varepsilon \to 0^+} \frac{1}{-p} \frac{\varepsilon^{p+1}}{\varepsilon} = 0.$$

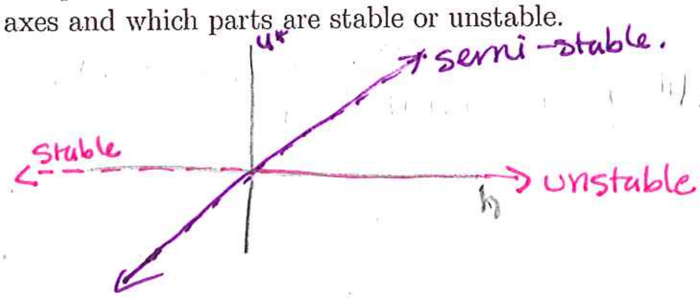
(c) Sketch a bifurcation diagram with respect to the parameter h, for the autonomous ODE $u' = u(h-u)$. Label your axes and which parts are stable or unstable.

$u^* = 0, h$

$F'(u) = h - u$

$F'(0) = h \to \begin{cases} \text{stable when } h<0 \\ \text{unstable otherwise} \end{cases}$

$F'(h) = 0 \to$ semistable $\forall h$.



(d) Use dominant balancing to rewrite the polynomial

$$\epsilon x^3 + x = 2, \qquad 0 < \epsilon \ll 1$$

so that you can use regular perturbation to find an expansion of the roots.

let $w = x/\delta$

$\frac{\varepsilon}{\delta^3} w^3 + \frac{w}{\delta} = 2$

mak $\frac{\varepsilon}{\delta^3} \sim \frac{1}{\delta} \to \delta = \sqrt{\varepsilon}$

re-write

$\frac{\varepsilon}{\varepsilon^{3/2}} w^3 + \frac{w}{\sqrt{\varepsilon}} = 2$

$w^3 + w = 2\sqrt{\varepsilon}$

(e) **BONUS** Find a two term approximation to the roots of the polynomial in part (d).

1st root

$X_\bullet = 2 - 8\varepsilon$

let $w = w_0 + \varepsilon^{\alpha} w_1 + \ldots$ take $\alpha = 1/2$

$w_0(w^2+1)$

$(w_0 + \varepsilon^{1/2} w_1 + \ldots)^3 + w_0 + \varepsilon^{1/2} w_1 = 2\sqrt{\varepsilon}$

$\varepsilon^0: \; w_0^3 + w_0 = 0 \;\to\; w_0 = 0, \pm i$ TOSS.

$\varepsilon^{1/2}: \; 3w_0^2 w_1 + w_1 = 2 \;\to\; w_1 = \frac{2}{1+3w_0} = -1$