MATH 46 WORKSHEET : deriving Green's Identities

Barnett 5/21/08

- first we want 'product rule' for div, ie what is $\vec{\nabla}\cdot(u\vec{J})$

scalar field. vector field

Apply usual product rule and gather terms back into vector notation (hint: look for $\vec{\nabla}u$)

$\vec{\nabla}\cdot(u\vec{J}) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\cdot\left(uJ_1, uJ_2\right) = \cdots$

- Write the above out for the case $\vec{J} = \vec{\nabla}v$: (v is some scalar field).

- Integrate over $\int_\Omega d\vec{x}$ then use Divergence Thm:

- You should get Green's 1st Identity: $\int_\Omega (u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v)\,d\vec{x} = \ ?$

- From this identity, subtract the identity with $u \leftrightarrow v$ swapped. This is Green's 2nd Identity:

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· ~ SOLUTIONS ~ ·

- first we want 'product rule' for div, ie what is $\vec{\nabla}\cdot(u\vec{J})$ (u: scalar field; $\vec{J}$: vector field)

Apply usual product rule and gather terms back into vector notation (hint: look for $\vec{\nabla}u$)

$$\vec{\nabla}\cdot(u\vec{J}) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\cdot(uJ_1, uJ_2) = \frac{\partial}{\partial x_1}(uJ_1) + \frac{\partial}{\partial x_2}(uJ_2)$$

$$= \frac{\partial u}{\partial x_1}J_1 + u\frac{\partial J_1}{\partial x_1} + \frac{\partial u}{\partial x_2}J_2 + u\frac{\partial J_2}{\partial x_2}$$

$$= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)\cdot(J_1, J_2) + u\left(\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2}\right)$$

$$= \vec{\nabla}u\cdot\vec{J} + u\vec{\nabla}\cdot\vec{J}$$

- Write the above out for the case $\vec{J} = \vec{\nabla}v$: (v is some scalar field).

$$\vec{\nabla}\cdot(u\vec{\nabla}v) = \vec{\nabla}u\cdot\vec{\nabla}v + u\underbrace{\vec{\nabla}\cdot\vec{\nabla}v}_{=\Delta v}$$

- Integrate over $\int_\Omega d\vec{x}$ then use Divergence Thm:

$$\int_\Omega \vec{\nabla}\cdot(u\vec{\nabla}v)$$

Div. Thm.

$$\int_{\partial\Omega} u\underbrace{\vec{\nabla}v\cdot\hat{n}}_{\frac{\partial v}{\partial n}}\,dA = \int_\Omega u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v\,d\vec{x}$$

- You should get Green's 1st Identity: $\int_\Omega (u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v)\,d\vec{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dA$

- From this identity, subtract the identity with $u \leftrightarrow v$ swapped. This is Green's 2nd Identity:

$$\int_\Omega (u\Delta v - v\Delta u)\,d\vec{x} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dA.$$