Math 46: X hour of 5/12/11: Degenerate Fredholm Equations

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We used Section 4.3.3, particularly Thms 4.12 and 4.13, to determine if the following had a solution, and then solve them. We made use of (4.31), the starred equation in lecture, a lot to get $u(x)$ once the $\mathbf{c}$ vector was found.

Let K operator have kernel $k(x,y) = \sin x \sin y$, on the interval $[0,\pi]$

First: Find the eigenvalues and eigenfunctions of K:

[Your eigenspaces are orthogonal, which is unusual. What property of K caused this?]

Use this to solve the following:

1. $Ku - u = \sin 2x$
2. $Ku - u = x$
   (We can use Maple to get the integrals $\int_0^\pi x \sin(nx) dx = \pi(-1)^{n+1}/n$)
3. $Ku = 3\sin 2x$
4. $Ku = 3\sin x$.
5. $Ku - \frac{\pi}{2}u = x$
6. $Ku - \frac{\pi}{2}u = \sin 3x$

Answer key:

A is 1-by-1 matrix with entry $\pi/2$. Spectrum of K is then $\pi/2$ (multiplicity 1, eigenfunction $\sin x$), and 0 (infinite multiplicity, eigenspace all functions orthog to $\{\beta_j\}$, i.e. orthog to $\sin x$)

Orthogonality of eigenspaces with different eigenvalues occurred since K was symmetric. Note not all degenerate kernels are symmetric.

1. $c_1 = 0$ so $u = -\sin 2x$
2. $c_1 = \frac{\pi}{\pi/2-1}$ so $u = \frac{\pi}{\pi/2-1} \sin x \; - \; x$
3. No solution, RHS not in span of α's.
4. Infinitely-nonunique solution, constrained only by $(\sin x, u) = 3$. Then $u =$ any particular solution + all hogomogeneous solutions (to $Ku = 0$). Eg, $u = \frac{6}{\pi} \sin x +$ (any function orthogonal to $\sin x$). May equally well write as $u = \frac{3}{2} +$ (any function orthogonal to $\sin x$).
5. λ is an eigenvalue. No solution since $f_1 \neq 0$.
6. $f_1 = 0$ so consistent, $c_1 =$ anything. So, $u(x) = \frac{2}{\pi}(-\sin 3x + c \sin x)$, with c anything.