5/7/08
Barnett.

MATH 46 WORKSHEET : Image (de)convolution in 1d

Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y) f(y)\,dy$, $k(s)$ = even symm, 2π-periodic
'aperture func.'

A) Show that $\phi_0(x) = 1$ is an eigenfunction of K, and find its eigenvalue λ_0:
[Hint: why is $K\phi_0(x)$ indep. of x? why is λ_0 indep. of x?]

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\cdots$ is eigenfunc. of K, find its eigenvalue λ_n:
[Hint: addition formula, k even]

C) How do λ_n relate to Fourier cos coeffs k_n of aperture func $k(s)$?

You could check that $\sin nx$ is also eigenfunc. w/ same eigenval. λ_n.

Assume image is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$
& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]$

D) How are g's Fourier coeffs related to those of f?

Such is the nature of convolution kernels. How would you invert $g \to f$, ie deconvolve?

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~ SOLUTIONS ~

Consider symm blurring operator $Kf(x) = \int_{-\pi}^{\pi} k(x-y) f(y)\, dy$, $k(s)$ = even symm 2π-periodic 'aperture func.'

A) Show that $\phi_0(x) = 1$ is an eigenfunction of K, and find its eigenvalue λ_0.
[Hint: why is $K\phi_0(x)$ indep. of x? why is λ_0 indep. of x?]

$$(K1)(x) = \int_{-\pi}^{\pi} k(x-y) \cdot 1 \cdot dy \underset{\text{change var!}}{\overset{s=y-x}{=}} \int_{-\pi-x}^{\pi-x} k(-s)\, ds \overset{\text{periodicity \& even-symm}}{=} \int_{-\pi}^{\pi} k(s)\, ds.$$ obviously const wrt. x.

so λ_0 = this const $= \int_{-\pi}^{\pi} k(s)\, ds$

B) Show that $\phi_n(x) = \cos nx$, $n=1,2,\cdots$ is eigenfunc. of K, find its eigenvalue λ_n:
[Hint: addition formula, k even]

$$(K\phi_n)(x) = \int_{-\pi}^{\pi} k(x-y) \cos ny\, dy = \int_{-\pi-x}^{+\pi-x} k(-s) \cos n(s+x)\, ds$$

periodic so can shift limit to $[-\pi, \pi]$.

bring out

$$= \int_{-\pi}^{\pi} k(s) \cos ns \cos nx\, ds - \underbrace{\int_{-\pi}^{\pi} k(s) \sin ns \sin nx\, ds}_{\text{zero since } k \text{ even symm}} = \underbrace{\cos nx}_{\phi_n(x)} \cdot \underbrace{\int_{-\pi}^{\pi} k(s) \cos ns\, ds}_{=\lambda_n \text{ eigenval}}$$

bring out

C) How do λ_n relate to Fourier cos coeffs k_n of aperture func $k(s)$? $\boxed{\lambda_n = \pi k_n}$

Since $k_n = \frac{1}{\pi}\int_{-\pi}^{\pi} k(s) \cos ns\, ds$ Euler-Fourier, or projection, formula.

You could check that $\sin nx$ is also eigenfunc. w/ same eigenval. λ_n.

Assume image is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

& after blurring $Kf(x) = g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]$

D) How are g's Fourier coeffs related to those of f?

Fourier basis = eigenbasis for K, so action of K is multiplication in this basis:

$$A_0 = \lambda_0 a_0 = \pi k_0 a_0$$

for $n=1,2,\cdots$
$$A_n = \lambda_n a_n = \pi k_n a_n$$
$$B_n = \lambda_n b_n = \pi k_n b_n$$

so Fourier coeffs get multiplied by $\pi \times$ Fourier coeffs of aperture func

Such is the nature of convolution kernels. How would you invert $g \to f$, ie deconvolve? divide Fourier coeffs by πk_n.