MATH 46 ① Barnett 5/15/07

Integral Equations

$\boxed{\text{Volterra}}$ operator $Ku(x) = \int_0^x k(x,y)\,u(y)\,dy$ 'lower-triangular' kernel

- $u - \lambda K u = f$ has <u>unique</u> solution for any λ, continuous func. f.
  This is $u = (1 + \lambda K + \lambda^2 K^2 + \cdots) f$. Neumann Series (converges)
- this tells you K has no eigenvalues
- The other way to solve a Volterra eqn. is to take derivs. via Leibniz's formula until you get an ODE, then solve that, with ICs that can be extracted from the $x \to 0$ limit of the integral eqn.

- You can go backwards, ie given an ODE convert to Volterra eqn. Make sure you can do this for 1st order, and for 2nd order using Lemma $\int_0^t \int_0^s f(r)\,dr\,ds = \int_0^t (t-s) f(s)\,ds$
- Volterra eqns arise in real-world situations where $u(t)$ determined by <u>history</u> $u(s)$ for $s<t$.

$\boxed{\text{Fredholm degenerate}}$ op. $Ku(x) = \sum_{j=1}^{N} \alpha_j(x) \beta_j(y)$ $\{\alpha_j\}$ L.I. set, $\{\beta_j\}$ " "

K need not be symm.

- Eigenvalues are those of matrix A with entries $A_{ij} = (\beta_i, \alpha_j)$,
  plus an ∞-multiplicity zero eigenvalue.
  Eigenfuncs are $\sum_{j=1}^{N} c_j \alpha_j(x)$ where $\vec{c}$ is corresponding eigenvector of A,
  plus the set of all funcs orthogonal to all $\{\beta_j\}$ forms the zero eigenspace

- $Ku - \lambda u = f$ has unique soln if $\lambda \neq$ eigenvalue,
  which can be got from $\sum_{j=1}^{N} \alpha_j(x)\, c_j - \lambda u(x) = f(x)$ $(*)$
  But if $\lambda = j^{th}$ eigenvalue then no soln. unless $\vec{f} = \{f_i\}$ is in the range of $A\vec{c} - \lambda\vec{c}$, ie $A\vec{c} - \lambda\vec{c} = \vec{f}$ consistent. $f_i = (\beta_i, f)$
- $Ku = f$ has no soln. unless f is in $\mathrm{Span}\{\alpha_j\}$ ← the range of K.
  when soln is nonunique in the zero eigenspace component.

$\boxed{\text{Fredholm symmetric}}$ op. $Ku(x) = \int_a^b k(x,y)\, u(y)\, dy$

with $k(y,x) = k(x,y)$ continuous.

so $(Ku, v) = (u, Kv)$ $\forall u,v \in L^2$.

- Eigenvalues λ_j real, tend to zero, ∞ number of them.
  Eigenfunctions ϕ_j orthogonal, complete in L^2 ... means form a basis for L^2.

All your techniques from symmetric matrices work:

- $Ku - \lambda u = f$ write $u = \sum_{j=1}^{\infty} c_j \phi_j$, $f = \sum_{j=1}^{\infty} f_j \phi_j$

  you may need to compute.

  gives $c_j = \dfrac{f_j}{\lambda_j - \lambda}$ by orthogonality

  ↳ tells you: unique soln. if $\lambda \neq$ eigenvalue

  otherwise nonunique solution (if $\lambda = \lambda_j$ for some j): $u(x) = \underset{\text{arbitrary.}}{c}\, \phi_j(x) + \sum_{i \neq j} \dfrac{f_i\, \phi_i(x)}{\lambda_i - \lambda}$ if $f_i = 0$.

  or no solution if $f_i \neq 0$.

- This all applies for $\lambda = 0$ too.
- K operator can be written in spectral form $K = \sum_{j=1}^{\infty} \lambda_j \underbrace{\phi_j (\phi_j, \cdot)}_{\text{projection onto } j^{\text{th}} \text{ eigenfunc.}}$

  Equivalent to diagonalizing a symm. matrix.