Math 46, Applied Math (Spring 2011): Midterm 2

2 hours, 50 points total, 6 questions. Heed the available numbers of points. Good luck!

1. [6 points] Use integration by parts to find a 2-term asymptotic expansion for $I(\varepsilon) = \int_0^\varepsilon e^{-1/t} dt$ in the *small* parameter $\varepsilon \to 0^+$.

[BONUS: prove that the remainder term satisfies the needed condition for an asymptotic expansion]

2. [6 points] Write the first 3 terms (i.e. trivial term plus two more) in the Neumann series for the solution of

$$u(t) = 12t^2 + \lambda \int_0^t (t-s)u(s)\,ds \ ,$$

where $\lambda \in \mathbb{R}$ is some constant.

3. [9 points] Consider the integral operator $(Ku)(x) := \int_0^\pi \sin 2x \sin y \, u(y)dy$ acting on functions on $(0, \pi)$.

   (a) Give the general solution to $Ku(x) - 3u(x) = \sin x$, or explain why not possible.

(b) Give the general solution to $Ku(x) = \sin x$, or explain why not possible.

(c) Give the general solution to $Ku(x) = \sin 2x$, or explain why not possible .

(d) What are all eigenvalue(s) (with multiplicity) and eigenspace(s) of this operator?

4. [10 points]

(a) By converting to a Sturm-Liouville problem, find all positive eigenvalues and eigenfunctions of the operator K which acts as $(Ku)(x) := \int_0^1 k(x,y)u(y)dy$, with kernel

$$k(x,y) = \begin{cases} 1-x, & y < x \\ 1-y, & y > x \end{cases}$$

[Hint: you'll need to extract a boundary condition at each end.]

(b) Use the energy method on the SLP to show that there are no negative or zero eigenvalues. [If you couldn't get an SLP above, just demonstrate the energy method on the simplest SLP you can think of.]

5. [10 points]

A 1D 2π-periodic image f is blurred by applying a Fredholm operator K with convolution kernel $k(x, y) = k(x - y)$, with even, 2π-periodic aperture function

$$k(s) = -\ln\left(2\sin\left|\frac{s}{2}\right|\right) = \cos s + \frac{1}{2}\cos 2s + \frac{1}{3}\cos 3s + \cdots$$

Recall that such an operator has eigenvalues $\lambda_n = \pi k_n$, $n = 0, 1, \ldots$, where k_n are the Fourier cosine coefficients of $k(s)$.

(a) Given the image $f(x) = \sin 7x$ find the blurred image $g(x) = (Kf)(x)$:

(b) Give a formula for the Fourier coefficients $(\hat{a}_n, \hat{b}_n)$ of the best reconstructed image $\hat{f}$ given those (A_n, B_n) of a measured blurry image g. Can all Fourier coefficients be reconstructed? (explain; you may assume no noise here)

(c) If noise of size 10^{-3} pollutes each Fourier coefficient of g, and a noise of size 0.1 (ie 10%) is acceptable in $\hat{f}$, how many coefficients should be reconstructed?

(d) The aperture function is unbounded, $\lim_{s\to 0} k(s) = \infty$. Is the aperture function in $L^2([-\pi, \pi])$? Prove it.

6. [9 points] Short-answer questions.

(a) Let K be a symmetric Fredholm operator with eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Either give the general solution to $Ku - \lambda_1 u = \phi_2$ or explain why not possible.

(b) Let $\{f_n\}$ be a complete orthogonal set. Prove that no non-trivial function g can be added to the set whilst maintaining orthogonality of the resulting set.

(c) Give an example of an interval and a sequence of functions that converge in L^2 but not uniformly on that interval (sketching may help.)

[BONUS: give an example as in (c) but the other way round, i.e. uniform but not L^2]

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z \to \infty} \mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i \sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Addition formulae

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

Power-reduction identities

$$\begin{aligned} \cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\ \cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\ \cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\ \sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta) \end{aligned}$$

Leibniz's formula

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t) dt \;=\; \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t) dt \;-\; a'(x) f(x, a(x)) \;+\; b'(x) f(x, b(x))$$