Math 46, Applied Math (Spring 2011): Final

3 hours, 80 points total, 10 questions (worth between 5 and 9 points each). Good luck!

1. [9 points] Consider the Dirichlet eigenvalue problem for $y(x)$ in the interval $x \in (1, e)$,

$$-x^2 y'' = \lambda y, \qquad y(1) = y(e) = 0 \ .$$

   (a) Prove that any eigenvalues λ have a definite sign (which?)

   (b) Find WKB approximations to the nth eigenvalue λ_n and corresponding eigenfunction $y_n(x)$.

(c) Sketch an eigenfunction of very large eigenvalue magnitude, showing how frequency and amplitude change vs x.

[BONUS] In this particular problem, what are the accuracies of λ_n and $y_n(x)$?

2. [6 points] The radius r of the early phase of a nuclear fireball explosion is assumed to depend only on time t, the total energy released e (units ML^2T^{-2}), and the density of the surrounding gas ρ. Following G. I. Taylor in the 1940's, fill a dimensions matrix, and deduce the most specific formula you can for how the radius depends on the other variables.

3. [9 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y + 4\varepsilon y y'^2 = 0, \qquad 0 < \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Use the Poincaré-Lindstedt method to give a 2-term approximation. [Hint: rescale $\tau = \omega t$ where ω is perturbed from the value 1. Don't forget to match initial conditions. Partial credit if you can only do regular perturbation; doing that will jog your memory anyway…]

(b) Discuss briefly any differences in the uniformity of the approximation, and the reason, if regular perturbation theory were used instead.

4. [5 points] By converting into an ODE, find the unique solution $u(t)$ to the integral equation,

$$4\int_0^t (t-s)u(s)ds + u(t) = t, \qquad t > 0.$$

5. [9 points] Fourier & convolution stuff.

(a) Consider the 'step' function $H(x) = 1$ for $x > 0$, zero otherwise. Compute the convolution of H with itself.

(b) Find the Fourier transform of the function $u(x) = xe^{-x^2/2}$. [Hint: it's a derivative]

(c) Recall that in class you showed that the 'top hat' function $u(x) = 1$ for $0 < x < 1$ and zero otherwise, when convolved with itself, gives the continuous 'triangle hat' function,

$$T(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the Fourier transform of the function T.

6. [8 points] Consider the reaction-diffusion equation in $\Omega \subset \mathbb{R}^3$ with zero-flux boundary condition,

$$u_t = \Delta u - \alpha u, \quad \text{in } \Omega, \quad t > 0, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

where $\alpha(\mathbf{x})$ is a given spatially-dependent decay rate, and f a given initial distribution.

(a) Consider the simple case $\alpha(\mathbf{x}) = 0$. Prove that there is at most one solution.

(b) Adapt your proof to general $\alpha(\mathbf{x})$. What condition on $\alpha(\mathbf{x})$ enables your uniqueness proof to still work?

7. [9 points] K is a symmetric Fredholm integral operator acting on the domain $(0, \pi)$, with a complete set of eigenfunctions $\{\sin nx\}$ and eigenvalues $1/n^2$, labeled by $n = 1, 2, \ldots$.

(a) Find the *general* solution u to the equation $(Ku)(x) - u(x) = \sin x, \quad 0 < x < \pi$, or explain why it has no solution:

(b) Find the *general* solution u to the equation $(Ku)(x) - u(x) = \sin 2x, \quad 0 < x < \pi$, or explain why it has no solution:

(c) Find the *general* solution u to the equation $(Ku)(x) = \begin{cases} 1, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$, or explain why it has no solution:

[BONUS] Explain whether the solution u to part (c) is in $L^2[(0,\pi)]$:

8. [8 points] Electric potential u in an upper half-plane $x \in \mathbb{R}$, $y > 0$, filled with anisotropic medium, satisfies a PDE with a decay condition at infinity,

$$a^2 u_{xx} + u_{yy} = 0, \qquad u(x,0) = f(x),\ x \in \mathbb{R}, \qquad u(x,y) \text{ bounded as } y \to +\infty,$$

with $a > 0$ a given anisotropy constant, and f a given boundary voltage function.

(a) Is the PDE hyperbolic, parabolic, or elliptic?

(b) Use the Fourier transform method to derive the (unique) solution $u(x,y)$. Your answer should be in terms of a and f only. (Don't forget to explain where the $y \to +\infty$ condition enters.)

9. [8 points] Consider the Sturm-Liouville problem $-u'' = f(x)$ on the interval $0 < x < 1$, with mixed boundary conditions $\alpha u(0) + u'(0) = 0$, and $u(1) = 0$. Here α is some (Robin) constant.

(a) For fixed $\alpha \neq 1$, compute the Green's function for this SLP.

(b) Discuss as concretely as you can the solvability for general f, in the case when $\alpha = 1$.

10. [9 points] Short questions.

(a) Compute the outer solution (with its correct constant) for the perturbed BVP
$\varepsilon y'' + (x-2)y' + y = 0, \;\; y(0) = 1, \;\; y(1) = 0$, with $0 < \varepsilon \ll 1$.

(b) Is the sequence $f_n(x) = e^{-nx^2}$, $n = 1, 2, \ldots$, convergent to the zero function on $(-1, 1)$ pointwise? uniformly? in L^2?

(c) Is $10^9(e^x - 1 - x) = O(x^2)$ as $x \to 0$? Prove your answer.

(d) [BONUS] Recall that if K is any symmetric operator with a complete set of eigenfunctions, the kernel of K^{-1} may be written as an eigenfunction expansion. Derive instead an eigenfunction expansion for the kernel of K itself:

Useful formulae

Non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm\frac{1}{\varepsilon}\int k(x)dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty}\mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned} \cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\ \cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\ \cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\ \sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta) \end{aligned}$$

Leibniz's formula

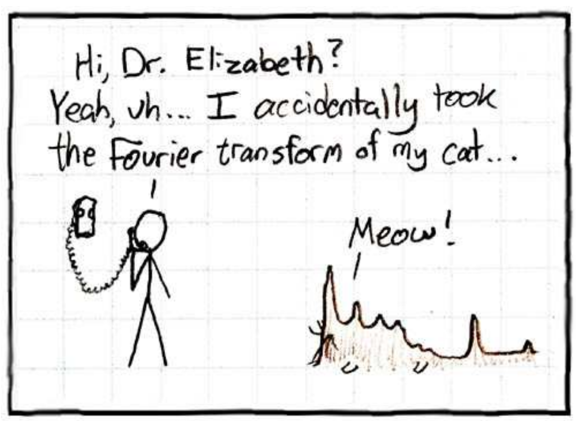
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt - a'(x)f(x,a(x)) + b'(x)f(x,b(x))$$

Fourier Transforms: $\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}u(x)dx$

$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{u}(\xi)d\xi$

$u(x)$	$\hat{u}(\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{ikx}	$2\pi\delta(k+\xi)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}$
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+\xi^2}$
$H(a-\lvert x\rvert)$	$2\frac{\sin(a\xi)}{\xi}$
$u^{(n)}(x)$	$(-i\xi)^n\hat{u}(\xi)$
$u * v$	$\hat{u}(\xi)\hat{v}(\xi)$

Here $H(x) = 1$ for $x \geq 0$, zero otherwise.



Greens first identity: $\int_\Omega u\Delta v + \nabla u\cdot\nabla v\, d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dA$

Product rule for divergence: $\nabla\cdot(u\mathbf{J}) = u\nabla\cdot\mathbf{J} + \mathbf{J}\cdot\nabla u$