Math 46, Applied Math (Spring 2009): Final

3 hours, 80 points total, 9 questions worth varying numbers of points

1. [8 points] Find an approximate solution to the following initial-value problem which is uniformly valid on $t > 0$ as $\varepsilon \to 0$, where $0 < \varepsilon \ll 1$ is a perturbation parameter.

$$\varepsilon y'' + 2ty' + ty = 0, \qquad y(0) = 2, \qquad \sqrt{\varepsilon} y'(0) = 1$$

(Be sure to present your answer purely in terms of the variables in the problem, and in a form without any integrals)

2. [9 points] Consider the Dirichlet eigenvalue problem on $0 < x < \pi$,

$$y'' = \lambda(1 + \sin x)^2 y, \qquad y(0) = y(\pi) = 0$$

(a) Prove that eigenvalues have a definite sign (which?)

(b) Find WKB approximations to the nth eigenvalue and corresponding eigenfunction.

(c) Sketch an eigenfunction with very large eigenvalue magnitude, showing how frequency and amplitude change vs x.

3. [9 points] Spread of pollutant concentration $u(\mathbf{x}, t)$ in an initially clean body of water $\Omega \subset \mathbb{R}^3$ obeys

$$u_t - \Delta u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, t > 0, \qquad \alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega$$

where f is the pollution source term, and $\alpha > 0$ a boundary absorption constant.

(a) Prove that a *steady-state* (time-independent) solution $u(\mathbf{x})$ to the PDE with given boundary conditions is unique. [Hint: set the t-derivative to zero]

(b) Prove that the time-dependent solution to the full equations above is unique

(c) The homogeneous steady-state case of the above is called a Stekloff eigenvalue problem with α as the eigenvalue:

$$\Delta u = 0 \quad \text{in } \Omega, \qquad\qquad \alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Prove that eigenfunctions from different eigenspaces are *orthogonal on the boundary*. [BONUS: prove α has a definite sign]

4. [7 points] In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = \text{const} \cdot l^\alpha$ when other parameters are held constant; give α.

5. [9 points] Bacterial evolution for times $t > 0$ can be modelled by the 1D reaction-diffusion equation in $x \in \mathbb{R}$,

$$u_t = u_{xx} + \alpha u, \qquad\qquad u(x,0) = f(x)$$

where α is a breeding/death rate constant.

(a) Use the Fourier transform method to write a general solution $u(x,t)$ for $t > 0$ in terms of the initial condition f and α.

(b) Fix $\alpha > 0$, i.e. positive breeding. What range of spatial frequencies ξ in the initial condition lead to exponential *growth* vs t (unstable as opposed to stable behavior)?

6. [7 points] Solve the following integral equation by converting to an ODE then solving (don't forget the boundary/initial conditions):

$$u(t) + \int_0^t (t-s)u(s)ds \; = \; t^2, \qquad t > 0$$

Must this solution be unique on each interval $0 < t < T$? If not, characterize the non-uniqueness, or, if so, explain what theorem proves your claim.

7. [10 points] Consider the Sturm-Liouville operator $Au := -u'' - \frac{1}{4}u$ on $[0, \pi]$ with Neumann boundary conditions $u'(0) = u'(\pi) = 0$.

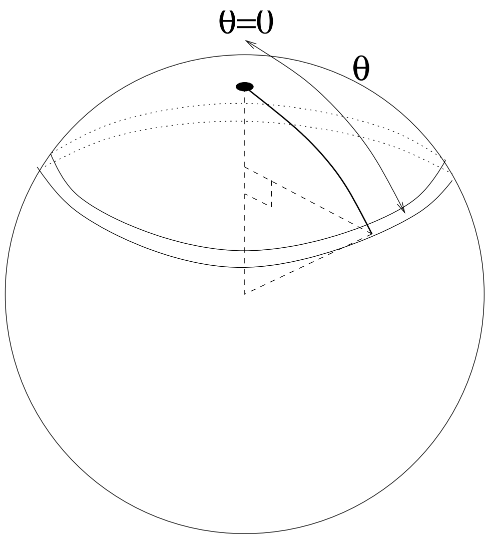
(a) Find the set of eigenfunctions and corresponding eigenvalues of A. (If you label by n, be sure to state whether counting starts at $n = 0$ or $n = 1$, etc)

(b) Does the equation $Au = f$ with the above boundary conditions have a Green's function? If so, find an expression for it; if not, explain in detail why not.

(c) Use the Green's function, or if not possible, another ODE solution method, to write an explicit formula for the solution $u(x)$ to $Au = f$ with the above boundary conditions, in terms of a general driving $f(x)$.

(d) [BONUS] What is the spectrum of the Green's operator $Gu(x) = \int_0^\pi g(x,\xi)u(\xi)d\xi$, or the solution operator you used above?

8. [7 points] Use the conservation law approach to derive the heat equation on the *surface* of the unit sphere for temperature distributions $u(\theta, t)$ which depend only on polar angle $0 < \theta < \pi$ as shown (and not on longitude), and on time t. As usual you may use Fick's Law that flux is $-k$ times the gradient of u. [Hint: remember you are working on a surface not in a volume. The diagram shows that the radius of the circle at polar angle θ is $\sin\theta$.]



[BONUS: find the general form of a solution to Laplace's equation on this sphere with the above symmetry]

9. [14 points] Short-answer questions

(a) Give an example of an interval and an infinite sequence of functions which are orthogonal on this interval but not complete.

(b) The *variance* of a probability distribution function $p(x)$ is defined as $\int_{-\infty}^{\infty} x^2 p(x) dx$. Find a formula for the variance as a certain derivative of the Fourier transform of p evaluated at a certain frequency.

(c) Let K be a *self-adjoint* operator with a complete set of orthogonal eigenfunctions. Prove that $Ku - \lambda u = f$ can only be solvable if f is orthogonal to all solutions v of the homogeneous problem $Kv - \lambda v = 0$.

(d) As $\lambda \to +\infty$, is $e^{-\lambda} = O(\lambda^{-n})$ for each $n = 0, 1, \ldots$? (Prove your answer)

(e) Place the following four terms in the *correct* order to form an asymptotic series as $\varepsilon \to 0$:

$$f(\varepsilon) \sim \varepsilon^{5/2} + \varepsilon^2 + \varepsilon^{-2} + \varepsilon^2 \ln \varepsilon + \ldots$$

(f) A 2π-periodic 1D image f is blurred by a symmetric convolution kernel to give g. Explain when and why it is sometimes impossible to reconstruct f from g.

[BONUS: Also explain the effect of the smoothness (differentiability) of this kernel on the ability to reconstruct f from a noisy measured data g]

Useful formulae

Non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x)dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty}\mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Leibniz's formula

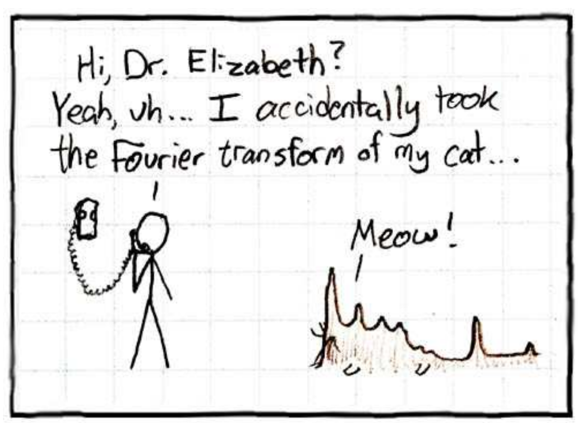
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt - a'(x)f(x,a(x)) + b'(x)f(x,b(x))$$

Fourier Transforms: $\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}u(x)dx$

$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{u}(\xi)d\xi$

$u(x)$	$\hat{u}(\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{ikx}	$2\pi\delta(k+\xi)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}$
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+\xi^2}$
$H(a-\lvert x\rvert)$	$2\frac{\sin(a\xi)}{\xi}$
$u^{(n)}(x)$	$(-i\xi)^n\hat{u}(\xi)$
$u * v$	$\hat{u}(\xi)\hat{v}(\xi)$

Here $H(x) = 1$ for $x \geq 0$, zero otherwise.



Greens first identity: $\int_\Omega u\Delta v + \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dA$

Product rule for divergence: $\nabla \cdot (u\mathbf{J}) = u\nabla\cdot\mathbf{J} + \mathbf{J}\cdot\nabla u$