MATH 46 WORKSHEET: Volterra integral equations

Barnett
W 4/30/08

- Differentiate to solve the following for the function $u(t)$. [Assume $f'$ exists and $a \neq 0$].

$$\int_0^t y\,u(y)\,dy \; - \; a\,u(t) \; = \; f(t) \qquad \text{on } 0 \le t \le 1.$$

[Hint: convert to ODE; what is the IC?]

- Prove the lemma.
$$\int_a^x \int_a^s f(y)\,dy\,ds \; = \; \int_a^x f(y)\,(x-y)\,dy$$

[Hint: define $F(s) := \int_a^s f(y)\,dy$
& write $\int_a^x F(s)\,ds$ as $\int_a^x \underbrace{1}_{u} \cdot \underbrace{F(s)}_{v'}\,ds$ ]

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— SOLUTIONS —

- Differentiate to solve the following for the function $u(t)$. [Assume $f'$ exists and $a \neq 0$]

this was #11 on p. 245

$$\int_0^t y\,u(y)\,dy - a\,u(t) = f(t) \qquad \text{on } 0 \le t \le 1$$

[Hint: convert to ODE; what is the IC?]

$$\Big\downarrow \frac{d}{dt}$$

$$t\,u(t) - a\,u'(t) = f'(t)$$

put in std 1st-order lin form: $\quad u' \underbrace{- \frac{t}{a}}_{p(t)} u = \underbrace{-\frac{f'(t)}{a}}_{g(t)}$ ODE

IC get by sub $t=0$ into integral eqn: $\cancel{\int_0^0 y\,u(y)\,dy} - a\,u(0) = f(0)$

$$\Rightarrow u(0) = -\frac{f(0)}{a}.$$

Solve IVP: integrating factor $\mu(t) = e^{\int p(t)dt} = e^{-\frac{t^2}{2a}}$

$$u = \frac{1}{\mu}\left[\int \mu g + c\right] = e^{t^2/2a}\left[\int e^{-\frac{t^2}{2a}} \frac{f'(t)}{-a} + c\right]$$

$$= -a\,e^{t^2/2a}\int_0^t e^{-s^2/2a} f'(s)\,ds + c\,e^{t^2/2a}$$

match ICs gives $\quad u(t) = -a\,e^{t^2/2a}\left[\int_0^t e^{-s^2/2a} f'(s)\,ds + f(0)\right]$

- Prove the lemma. $\quad \int_a^x \int_a^s f(y)\,dy\,ds = \int_a^x f(y)\,(x-y)\,dy$

[Hint: define $F(s) := \int_a^s f(y)dy$ & write $\int_a^x F(s)\,ds$ as $\int_a^x \underbrace{1}_{u'} \cdot \underbrace{F(s)}_{v} ds$ ]

sorry!

note $v' = f$ which is useful.

$$\int_a^x \overset{u'}{1} \cdot \overset{v}{F(s)}\,ds = -\int_a^x \overset{u}{s}\,\overset{v'}{f(s)}\,ds + \left[\overset{uv}{sF(s)}\right]_a^x$$

$$= -\int_a^x s f(s)\,ds + x\int_a^x f(s)\,ds$$

$$= \int_a^x f(s)\,(x-s)\,ds$$

or could replace s with y.