Math 46, Applied Math (Spring 2008): Midterm 1

2 hours, 50 points total, 5 questions worth varying number of points

1. [9 points] In modeling an atomic explosion, G. I. Taylor supposed there was a law relating the fireball radius r to time t after explosion, and the two fixed parameters E energy released (units: mass times speed squared) and ρ the initial air density.

   (a) How many independent dimensionless quantities are there? Give them.

   (b) From this deduce as much as you can about how r must scale with t.

   (c) If the law is enlarged to include dependence on an extra fixed parameter a, the acceleration due to gravity, use the Buckingham Pi Theorem to deduce whether with all three parameters fixed, the scaling of r with t must be as before.

2. [8 points] Find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - (1-x)^2 y' - y = 0, \qquad y(0) = y(1) = 1$$

where $0 < \varepsilon \ll 1$. [Hint: if you think an integral is difficult, it's not; just substitute].

3. [8 points] Consider the linear homogeneous ODE, $\qquad -y'' = \lambda(4x - x^2)^2 y, \qquad$ on $2 < x < 3$.

(a) For what λ is the problem oscillatory, or non-oscillatory, in character?

(b) Write down an approximate *general* solution to the ODE that is accurate for large *positive* λ.

(c) Use this to get an approximation for the sequence of values λ, *and* corresponding solutions $y(x)$, such that there is a nontrivial solution with boundary conditions $y(2) = 0$ and $y(3) = 0$. [Hint: use the lower boundary condition to make your life easier. Don't forget to write the solutions $y(x)$ too].

(d) [BONUS] Find the values λ if the boundary conditions are $y'(2) = 0$ and $y(3) = 0$.

4. [10 points] Short answer questions.

(a) Is $f(t, \lambda) = t^\lambda$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (0, 1)$, as $\lambda \to \infty$? (briefly explain).

(b) Does $e^{-t} = o(t^{-\alpha})$ hold as $t \to \infty$, for any fixed $\alpha > 0$? Prove your answer.

(c) Sketch the bifurcation diagram, in the domain $-1 \leq h \leq 1$, for the autonomous ODE $u' = u^2 + h^2 - 1$. Label your axes, and which parts are stable or unstable.

(d) What, if any, issues do you see in attempting singular perturbation in the problem $\varepsilon y'' + xy' + xy = 0$, $y(-1) = 2$, $y(1) = 3$, for $\varepsilon \ll 1$? (Do not solve the whole thing!)

(e) [BONUS] Modify the interval in part a) so that the uniform convergence property is changed, but not the pointwise convergence.

5. [15 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(y')^2, \qquad \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Find a 2-term asymptotic approximation using regular perturbation theory. [Hints: You may find the power-reduction identities on the last page useful. You will get partial credit for leaving the 2^{nd} term as the solution to a clearly-specified IVP.]

(b) Is this a uniform approximation for $t \in (0, \infty)$? Why?

(c) Use the Poincaré-Lindstedt method to give a more useful 2-term approximation. [Hint: rescale to $\tau = \omega t$ where ω is perturbed from the value 1]

(d) Is this a uniform approximation for $t \in (0, \infty)$?

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty} \mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta \sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta \sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$