Have a great summer!

Barnett 5/31/08

— SOLUTIONS —

# Math 46, Applied Math (Spring 2008): Final

— (unchecked for errors — look online for any corrections).

Take a deep breath, do the questions in any comfortable order for you. Enjoy!

3 hours, 80 points total, 9 questions, roughly in syllabus order (apart from short answers)

1. [16 points. Note part ~~c~~ d, worth 7 points, is independent of the others]
   A nonlinear damped oscillator is given by the initial-value problem

$$my'' + ay' + ky^3 = 0 \qquad y(0) = 0 \qquad my'(0) = I$$

(a) If $m$ is a mass, find the dimensions of the other three parameters $a, k, I$ (recall $y$ is a displacement, *i.e.* length).

$$\begin{array}{c c} & \begin{array}{cccc} m & a & k & I \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ & & -2 & 1 \\ & -1 & -2 & -1 \end{bmatrix} \end{array}$$

(b) Write down *two* length scales and *two* time scales.

$$y_c = \frac{a}{\sqrt{mk}}, \quad \frac{I}{a}, \quad \ldots$$

or anything of form $\frac{I}{a}\left(\frac{a^4}{kmI^2}\right)^{\alpha}$

$$t_c = \frac{m}{a}, \quad \frac{a^3}{kI^2}, \quad \ldots \quad \text{or} \quad \frac{m}{a}\left(\frac{a^4}{mkI^2}\right)^{\alpha}$$

$\alpha \in \mathbb{R}$

(c) Show that when the model is non-dimensionalized using scaling appropriate for the *small mass* limit (choose time and length scales which don't involve $m$), the IVP

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1$$

results. What is $\varepsilon$ in terms of the original parameters?

ie $y_c = \frac{I}{a}$, $t_c = \frac{a^3}{kI^2}$

rescale

$$m\frac{y_c}{t_c^2}\bar{y}'' + a\frac{y_c}{t_c}\bar{y}' + ky_c^3\bar{y} = 0, \quad \bar{y}(0) = 0, \quad m\frac{y_c}{t_c}\bar{y}'(0) = I$$

subst. $y_c, t_c$

$$\frac{mIk^2I^4}{a\,a^6}\bar{y}'' + \cancel{a}\frac{IkI^2}{\cancel{a}a^3}\bar{y}' + \frac{kI^3}{a^3}\bar{y} = 0, \qquad m\frac{I^3k}{a^4}\bar{y}'(0) = I$$

divide by $\frac{kI^3}{a^3}$

$$\underbrace{\frac{mkI^2}{a^4}}_{\varepsilon}\bar{y}'' + \bar{y}' + \bar{y}_1 = 0, \quad \bar{y}(0) = 0, \qquad \underbrace{\frac{mkI^2}{a^4}}_{\varepsilon}\bar{y}'(0) = 1$$

↗ uniform approximation

(d) Find a *leading-order* perturbation approximation to the solution of the IVP from (~~b~~ c), and give a crude sketch showing any key features. Here it is written out again:

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1 \qquad \varepsilon \ll 1$$

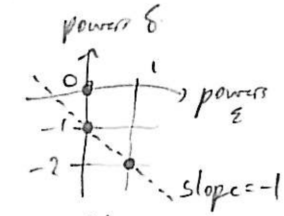
singular perturbation ($\varepsilon$ on highest deriv.) $\Rightarrow$ initial layer problem.

$\boxed{\text{inner:}}$ $\tau = \frac{t}{\delta}$ so $\underbrace{\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y'}_{\text{dominant balance}} + Y^3 = 0$

rescale.

mult. by $\varepsilon$

dominant balance so $\varepsilon = \delta$.




so $\delta^{-1}\varepsilon = 1$.

$$Y'' + Y' + \cancel{\varepsilon Y^3} = 0$$

↳ ignore at leading order.

ICs: $Y(0) = 0$

$\frac{\varepsilon}{\delta} Y'(0) = 1$, $\frac{\varepsilon}{\delta} \to 1$

Gen. soln. $Y_i(\tau) = A e^{-\tau} + B$

match to ICs: $Y_i(0) = 0$ gives $A + B = 0$

$Y_i'(0) = 1$ gives $A = -1$ so $Y_i(\tau) = 1 - e^{-\tau}$

$\lim_{\tau \to \infty} Y_i(\tau) = 1 = c_m$ = common limit, now use to fix the outer soln.

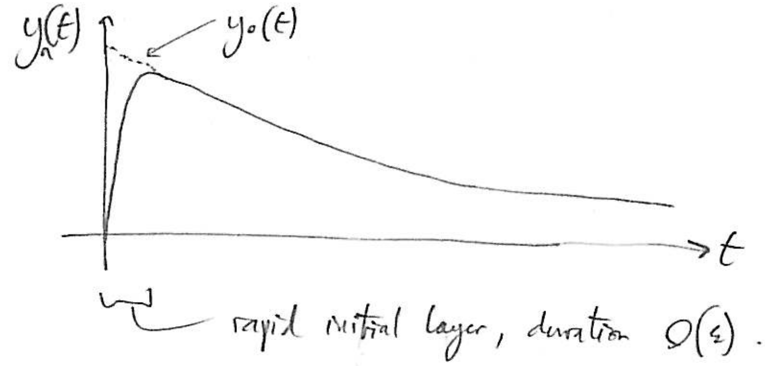
$\boxed{\text{outer:}}$ $y' + y^3 = 0$ gen soln. by integrating $\int \frac{dy}{y^3} = -\int dt$

$\Rightarrow -\frac{1}{2} y_0^{-2} = -t + c \Rightarrow y_0(t) = \frac{1}{\sqrt{2t + c}}$

$\lim_{t \to 0} y_0(t) = \frac{1}{\sqrt{c}} = c_m$ so $c = c_m^{-2} = 1$

Combine: $y_a(t) = y_0 + y_i - c_m = \frac{1}{\sqrt{2t+1}} - e^{-t/\varepsilon}$

Don't forget to sketch; you can do intuitively even without solving (1 point):




2. [6 points] Formulate the IVP

$$u'' + u' + tu = 1, \quad u(0) = 2, \quad u'(0) = 1$$

as a Volterra integral equation of the form $Ku - \lambda u = f$ (do not try to solve).

integrate wrt $t$. (once you've written using dummy variable $s$)

$$u'(t) - \underbrace{u'(0)}_{2} + u(t) - \underbrace{u(0)}_{1} + \int_0^t s\, u(s)\, ds = t$$

(integrate, again using dummy var. when necessary.

function in Lemma.

$$u(t) - \underbrace{u(0)}_{2} - 2t + \int_0^t u(s) ds - t + \underbrace{\int_0^t \int_0^s \boxed{r\, u(r)}\, dr\, ds}_{} = \frac{t^2}{2}$$

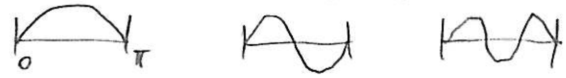
use Lemma, gives

$$\int_0^t (t-s)\, s\, u(s)\, ds$$

rearrange:

$$\underbrace{\int_0^t \big(1 + (t-s)s\big)\, u(s)\, ds}_{(Ku)(t)} + \underbrace{u(t)}_{\lambda = -1} = \underbrace{\frac{t^2}{2} + 3t + 2}_{f(t)}$$

$n=1,\ \lambda_1=\frac{1}{1}$ $\quad n=2\ \lambda_2=\frac{1}{2^2}$ $\quad n=3,\ \lambda_3=\frac{1}{3^2}$ etc.




3. [~~8~~ 9 points] $K$ is a symmetric Fredholm operator on $[0,\pi]$ with continuous kernel, a complete set of (unnormalized) eigenfunctions $\{\sin nx\}$ labeled by $n = 1, 2, \ldots$, with corresponding eigenvalues $1/n^2$.

3 (a) Use this to find the general solution to $Ku(x) - 2u(x) = \sin 2x$, or explain why not possible.

$\lambda \neq$ any eigenvalue, so soln. exists & is unique.

if write $u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)$ and $f(x) = \sum_{j=1}^{\infty} f_j \phi_j(x)$ then $c_j = \dfrac{f_j}{\lambda_j - \lambda}$ $\forall j$

Here $f(x) = \sin 2x$ so $f_2 = 1,\ f_j = 0\ j \neq 2$, so $c_2 = \dfrac{1}{\frac{1}{2^2} - 2} = -\dfrac{4}{7}$

so $u(x) = -\frac{4}{7} \sin 2x$

2 (b) Find the general solution to $Ku(x) - \frac{1}{4}u(x) = \sin 2x$, or explain why not possible.

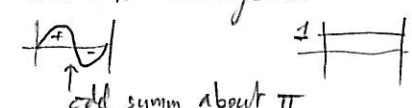
now $\lambda = \lambda_2 = \frac{1}{4}$ so either no soln. or $\infty$ of solns

again, $\forall j$ $\quad (\lambda_j - \lambda) c_j = f_j$ $\quad$ but for $j=2$ get $(0) c_j = 2$ $\quad$ since same RHS as above

which has no solution.

4 (c) Find the general solution to $Ku(x) - \frac{1}{4}u(x) = 1$, or explain why not possible.

$\lambda$ same as above. But $f_2 = 0$ since $\sin 2x$ orthogonal to $1$




$\Rightarrow$ solution exists. but $c_2$ unknown $\quad$ Need other $f_j$'s

Euler-Fourier:

Fourier sine series of the function $1$: $\quad f_j = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} 1 \cdot \sin nx\, dx$

$$= \frac{2}{\pi}\frac{1}{n}\Big[-\cos nx\Big]_0^{\pi} = \frac{1 - \cos n\pi}{n\pi}$$

$$= \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Now use $c_j = \dfrac{f_j}{\lambda_j - \lambda}$:

$$u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) = \sum_{j=1,3,5\ldots}^{\infty} \frac{4/n\pi}{1/n^2 - 1/4} \sin nx + c_2 \sin 2x, \quad \text{for } c_2 \in \mathbb{R}$$

4. [9 points] The following PDE describes a chemical with concentration $u(\mathbf{x}, t)$ diffusing while being broken down by the environment at given rate $\alpha(\mathbf{x}) \geq 0$. Prove that any solution to the IVP is unique.

7

$$u_t = \Delta u - \alpha u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}) \quad \text{in } \Omega$$

Energy method: if $u_1$ & $u_2$ are solutions, then $u = u_1 - u_2$ obeys above PDE w/ BC, homogeneous ICs $\left(u(\vec{x},0) = 0 \text{ in } \Omega\right)$

Then, mult by $u$ & integrate

$$\underbrace{\int_\Omega u u_t \, d\vec{x}}_{\frac{1}{2}\frac{d}{dt}\boxed{\int_\Omega u^2 d\vec{x}}} = \underbrace{\int_\Omega u \Delta u \, d\vec{x}}_{\text{Greens } 1^{st} \text{ id.}} - \int_\Omega \alpha(\vec{x}) u^2 \, d\vec{x}$$

call $E(t)$

Greens $1^{st}$ id.: $-\int_\Omega \vec{\nabla} u \cdot \vec{\nabla} u \, d\vec{x} + \int_{\partial\Omega} \underbrace{u}_{0 \text{ by BCs}} \frac{\partial u}{\partial n} dA$

$$\text{so} \quad \frac{1}{2} E'(t) = -\underbrace{\int_\Omega |\vec{\nabla} u| \, d\vec{x}}_{\geq 0} - \underbrace{\int_\Omega \alpha(\vec{x}) u^2 d\vec{x}}_{\geq 0} \leq 0$$

since $\alpha(\vec{x}) \geq 0$ everywhere in $\Omega$.

But $E(t) \geq 0$ by definition, and at $t=0$, $E(0) = \int_\Omega u(\vec{x},0)^2 d\vec{x} = 0$ by homog. ICs.

So $E(t) = 0 \quad \forall t \geq 0$

$\Rightarrow \quad u(\vec{x}, t) = 0 \quad \forall t \geq 0$ and $\forall \vec{x} \in \Omega$, identically zero.

$\Rightarrow \quad u_1 \equiv u_2$, no two solutions can differ $\Rightarrow$ unique.

2

An extra drift term is added to the right-hand side of the PDE giving $u_t = \Delta u - \mathbf{c} \cdot \nabla u - \alpha u$. Find a condition on the drift velocity field $\mathbf{c}(\mathbf{x})$ such that your above proof method still works.

The extra term appears on RHS when do energy method:

$$\frac{1}{2} E'(t) = -\int_\Omega |\vec{\nabla} u|^2 d\vec{x} - \int_\Omega \alpha(\vec{x}) u^2 d\vec{x} - \int_\Omega u \vec{c} \cdot \vec{\nabla} u \, d\vec{x}$$

Proof still works if $I := \int_\Omega u \vec{c} \cdot \vec{\nabla} u \, d\vec{x}$ can be shown to be $\geq 0$ since then $E(t) \leq 0$ as before.

note $\vec{\nabla} \cdot (\vec{c} u^2) = u^2 \vec{\nabla} \cdot \vec{c} + \vec{c} \cdot \boxed{\vec{\nabla}(u^2)}$ $\quad \boxed{\vec{\nabla}(u^2)} \to 2u\vec{\nabla} u$

$$\text{So} \quad I = \underbrace{\frac{1}{2} \int_\Omega \vec{\nabla} \cdot (\vec{c} u^2) \, d\vec{x}}_{\text{Div thm} \to \int_{\partial\Omega} u^2 \vec{n} \cdot \vec{c} \, \to 0 \text{ by BCs}} - \frac{1}{2} \int_\Omega u^2 \vec{\nabla} \cdot \vec{c} \, d\vec{x} \geq 0 \quad \text{if} \quad \boxed{\vec{\nabla} \cdot \vec{c} \leq 0 \text{ everywhere in } \Omega}$$

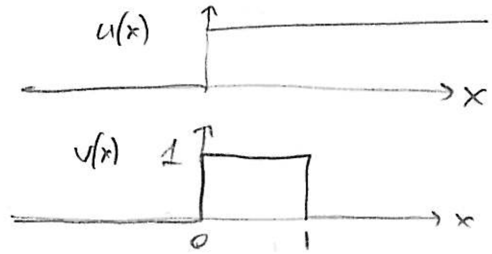
ie non-diverging (only contracting) flow!

people found this very hard, even though it is basic calculus:

5. [5 points] Compute directly the convolution $(u * v)(x)$ of the following two functions on $\mathbb{R}$:

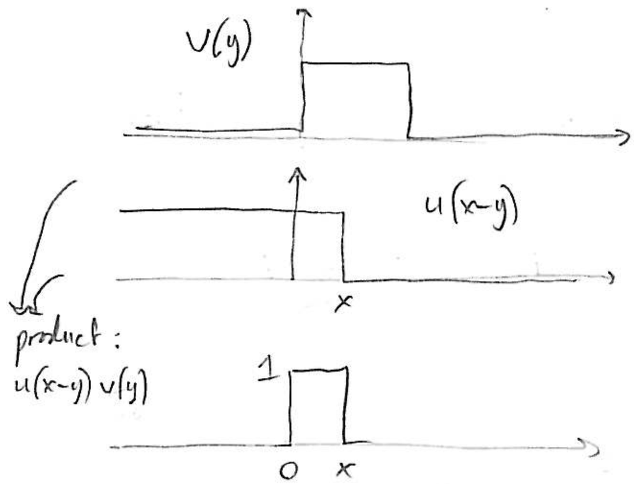
$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \text{(this is the unit step function),}$$

$$v(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(this is a top-hat function).}$$




$$(u * v)(x) = \int_{-\infty}^{\infty} u(x-y)\, v(y)\, dy$$

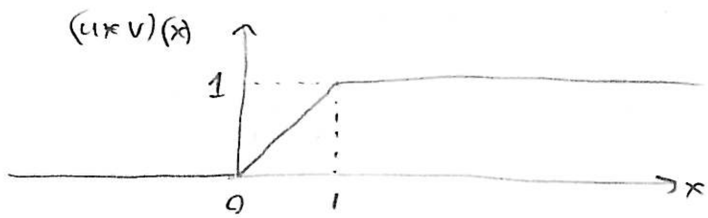
plot $u(x-y)$ by flipping left-right then translating forward by $x$.




← in the case $0 < x < 1$.

so for $\begin{cases} x \leq 0, & \text{integral is } 0 \\ 0 < x < 1, & \text{integral is } x \\ x \geq 1, & \text{integral is } 1 \end{cases}$

so $$(u*v)(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$




you can also do without graphs by using limits of integration

6. [7 points] Use Fourier transforms to compute the convolution of the Cauchy distribution function

$$u(x) = \frac{1}{1+x^2}$$

with itself.

Cauchy ↓

Notice $\frac{1}{1+x^2}$ looks like $e^{-a|\xi|} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \frac{2a}{a^2+\xi^2}$ in the FT table.

Need to flip around so Cauchy is in $x$ not $\xi$.

From this line of table, $e^{-a|x|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+\xi^2} e^{-ix\xi}\, d\xi$

exchange $x \leftrightarrow \xi$, take conjugate & divide by $2\pi$: $2\pi e^{-a|\xi|} = \int_{-\infty}^{\infty} \frac{2a}{a^2+x^2} e^{ix\xi}\, dx$ (*)

choosing $a=1$, $\mathcal{F}\left(\frac{1}{1+x^2}\right) = \pi e^{-|\xi|}$

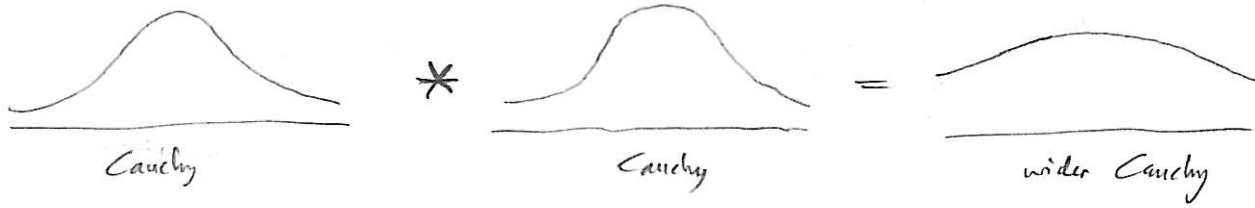
Convolution theorem: if $w = u * u$ then $\hat{w} = \hat{u}\hat{u}$

so $(u*u)(x) = \mathcal{F}^{-1}(\hat{u}^2)(x) = \mathcal{F}^{-1}(\pi^2 e^{-2|\xi|})$

But choosing $a=2$ in (*) gives $\mathcal{F}^{-1}(2\pi e^{-2|\xi|})(x) = \frac{2a}{a^2+x^2} = \frac{4}{4+x^2}$

$\Rightarrow (u*u)(x) = \frac{\pi}{2}\cdot\frac{4}{4+x^2} = \frac{2\pi}{4+x^2}$

[BONUS: describe precisely the linear transformation required to take the original Cauchy function to your above answer, and interpret].




Notice $u(x/2) = \frac{1}{1+\frac{x^2}{4}} = \frac{4}{4+x^2}$ so convolution has made the Cauchy twice as wide, and $\frac{2\pi}{4} = \pi/2$ times less tall.

What is amazing is that the distribution became twice as wide (for Gaussians the factor is only $\sqrt{2}$ as wide).

notice only the initial velocity is inhomogeneous. (IC for value is 0).

7. [~~6~~ 8 points] Use Fourier transforms to solve the 1D wave equation $u_{tt} = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$, with initial conditions $u(x, 0) = 0$, and $u_t(x, 0) = f(x)$ for a general function $f$. Try to give an answer involving a real-space integral. [Hint: after you use the ICs, combine things to make a trig function]

$$u_{tt} = u_{xx} \xrightarrow{\mathcal{F} \text{ in } x} \hat{u}_{tt}(\xi,t) = \underbrace{-\xi^2}_{\text{due to how } 2^{\text{nd}} \text{ deriv transforms.}} \hat{u}(\xi,t)$$

this is an ODE in $t$, for each fixed $\xi$, of $2^{\text{nd}}$ order const-coeff type ($\hat{u}'' + \xi^2 \hat{u} = 0$) makes oscillatory

Gen soln: $\hat{u}(\xi,t) = \hat{a}(\xi) \cos \xi t + \hat{b}(\xi) \sin \xi t$

note 'constants' here can depend on $\xi$.

Match ICs (FT of): @ $t=0$

$$\hat{u}(\xi,0) = 0 \Rightarrow \hat{a}(\xi) = 0 \quad \forall \xi \in \mathbb{R}$$

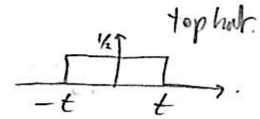
$$\hat{u}_t(\xi,0) = \xi \hat{b}(\xi) \underbrace{\cos \xi t \Big|_{t=0}}_{1} = \hat{f}(\xi) \quad \forall \xi \in \mathbb{R}$$

$$\Rightarrow \hat{b}(\xi) = \frac{\hat{f}(\xi)}{\xi}$$

use convolution to handle this product.

$$\Rightarrow \text{Soln. is } \hat{u}(\xi,t) = \frac{\hat{f}(\xi)}{\xi} \sin \xi t = \hat{f}(\xi) \cdot \frac{\sin \xi t}{\xi}$$

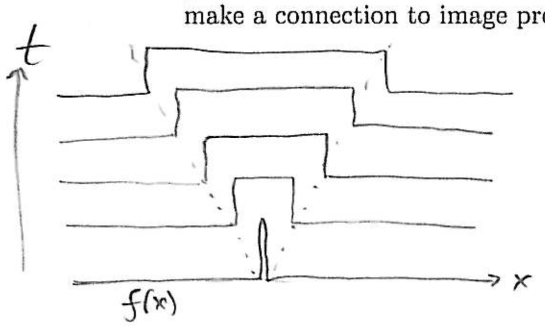
$\frac{1}{2} H(t-|x|)$ by the FT table, $5^{\text{th}}$ row:




$$\rightarrow u(x,t) = f * \mathcal{F}^{-1}\left(\frac{\sin \xi t}{\xi}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} H(t - |x-y|) f(y)\, dy = \frac{1}{2} \int_{x-t}^{x+t} f(y)\, dy$$

[BONUS: describe in words the action that propagation in time has upon the initial function, and make a connection to image processing].




the solution at time $t>0$ is the blurring of the image $f(x)$ by the tophat aperture function of height $\frac{1}{2}$ and half-width $t$.

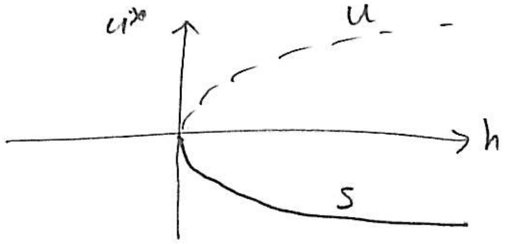
Propagating in time is convolution by an aperture widening at speed 1! (the wave speed).

(you could deblur to recover $f(x)$ from $u(x,t)$).

9. [10 points] More short answers!

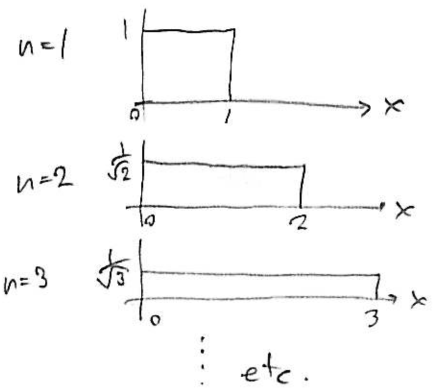
3 (a) Sketch a bifurcation diagram, including stability, for the autonomous ODE $u' = u^2 - h$. (h is the parameter)

$f(u) = u^2 - h = 0$ when $\begin{cases} u^* = \pm\sqrt{h} & \text{for } h \geq 0 \\ \text{no real values} & \text{for } h < 0 \end{cases}$




$f'(u) = 2u$ so $u^* > 0$ gives unstable, $u^* < 0$ stable.

3 (b) In the limit $n \to +\infty$ does the top-hat sequence $f_n(x) = n^{-1/2}$ for $x < n$, zero otherwise, converge to the zero function on $[0, \infty)$ pointwise? uniformly? in $L^2$ sense? (three binary answers required)




at every $x \in [0, \infty)$, $\lim_{n\to\infty} f_n(x) = 0 \Rightarrow$ pointwise ✓

for each $n$, $\max_{0 \leq x < \infty} f_n(x) = n^{-1/2}$ ('top' of hat).

$\to 0$ as $n \to \infty$

$\Rightarrow$ uniformly ✓

$\|f_n\| = \sqrt{\int_0^\infty f_n^2(x)dx} = \sqrt{\int_0^n n^{-1}dx} = \sqrt{1} \not\to 0$ as $n \to \infty$

$\Rightarrow$ not $L^2$ ✗

2 (c) Define *completeness* for a set of functions $\{\phi_j\}_{j=1,2,\ldots}$ on an interval $[a, b]$.

if $\{\phi_j\}$ is complete, then

$$(f, \phi_j) = 0 \quad \forall j = 1, 2, \ldots \quad \Rightarrow \quad f \equiv 0 \quad \text{(the zero function in } L^2\text{)}$$

(this is like spanning a vector space in the finite-dim case from Math 22).

2 (d) The *auto-correlation* of a (complex-valued) function $u(x)$ is defined as $C(x) = \int_{-\infty}^{\infty} \overline{u(y-x)}u(y)dy$ (note there is no typo, and bar means complex conjugate), and is useful in signal processing. Find its Fourier transform $\hat{C}(\xi)$ in terms of $\hat{u}(\xi)$. (This is called the Wiener-Khintchine theorem).

note $C(x) = (v * u)(x)$ where $v(x) = \overline{u(-x)}$ [careful: since $y-x$ not $x-y$, $C \neq \bar{u} * u$]

Convolution thm.:

$\hat{C}(\xi) = \hat{v}(\xi)\hat{u}(\xi)$ but $\hat{v}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} v(x)\,dx = \int_{-\infty}^{\infty} e^{ix\xi}\,\overline{u(-x)}\,dx$

so $\overline{\hat{v}(\xi)} = \int_{-\infty}^{\infty} e^{-ix\xi} u(-x)dx = \int_{-\infty}^{\infty} e^{ix\xi} u(x)dx = \hat{u}(\xi)$

change var to $-x$.

so $\hat{C}(\xi) = \overline{\hat{u}(\xi)}\,\hat{u}(\xi) = |\hat{u}(\xi)|^2$

This was tricky.

8. [10 points] Short answers.

$au_{xx} + bu_{xy} + cu_{yy}.$

2 (a) Is the PDE $u_{xx} + u_{yy} = 4u_{xy}$ parabolic, hyperbolic or elliptic?

convert to algebraic curve ... or discriminant-like form

$x^2 - 4xy + y^2 = 1$

has $a=1,\ b=-4,\ c=1.$

$\Rightarrow b^2 - 4ac = 16 - 4 = 12 > 0$

so hyperbolic.

can check for small $y$ there's no real soln $x$, $\Rightarrow$ hyperbolae

3 (b) Find the *general* solution to the PDE $u_{xy} = 1$ for $x, y \in \mathbb{R}$.

$$u_{xy} = 1$$

$$\int dx:\quad u_y = x + f(y)$$

$$\int dy:\quad u = xy + yf(y) + g(x)$$

$$u(x,y) = xy + h(y) + g(x).$$

3 (c) The speed $c$ of sound in a gas depends only on density $\rho$ and pressure $P$ (dimensions $ML^{-1}T^{-2}$). Deduce as much as you can about their relationship.

$$\begin{array}{c|ccc} & c & \rho & P \\ \hline M & & 1 & 1 \\ L & 1 & -3 & -1 \\ T & -1 & & -2 \end{array}$$

$\Pi_1 = \dfrac{P}{c^2\rho}$ is only 1 dimensionless quantity.

$\Rightarrow$ Buckingham Pi Thm tells us $F(\Pi_1) = 0$

ie $\Pi_1 = \text{const} \Rightarrow c = k\sqrt{\dfrac{P}{\rho}}$ (for adiabatic gas const is $5/3$ !)

2 (d) Use the Cauchy-Schwarz inequality to give an upper bound to the number $\int_0^1 y^2 f(y)dy$ in terms of $\|f\|$ on the interval $(0,1)$.

$$|(f,g)| \le \|f\|\,\|g\| \qquad \text{use } g(y) = y^2$$

$$\Big|\underbrace{\int_0^1 y^2 f(y)\,dy}_{(f,g)}\Big| \le \|f\| \cdot \|y^2\| \qquad \|y^2\| = \sqrt{\int_0^1 (y^2)^2\,dy} = \frac{1}{\sqrt{5}}$$

so an upper bound to its magnitude is $\frac{1}{\sqrt{5}}\|f\|$

(e) [BONUS] The $\boxed{\text{2-norm of an operator}}$ ($\|K\|$) is defined as $\max_{f\neq 0} \|Kf\|/\|f\|$. Compute the 2-norm of the Fredholm operator with kernel $xy$ on the interval $[0,1]$.

$$(Kf)(x) = \int_0^1 xy\, f(y)\,dy = x\int_0^1 y f(y)\,dy$$

using method of d), its size bounded by $\|f\|/\sqrt{3}$, and bound reached by $f(y) = y$

$$\Rightarrow \|Kf\| = \Big|\underbrace{\int_0^1 y f(y)\,dy}_{\text{a number}}\Big|\ \underbrace{\|x\|}_{\text{a function}} \le \frac{\|f\|}{\sqrt{3}} \cdot \|x\| = \frac{\|f\|}{3}$$

$$\|x\| = \sqrt{\int_0^1 x^2\,dx} = \frac{1}{\sqrt{3}}$$

which is tight by above,

So $\|K\| = \frac{1}{3}$.

## Useful formulae

Non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm\frac{1}{\epsilon}\int k(x)dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\operatorname{erf}(0) = 0$ and $\lim_{z\to\infty}\operatorname{erf}(z) = 1$]:

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta)\\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta)\\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta)\\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Leibniz's formula

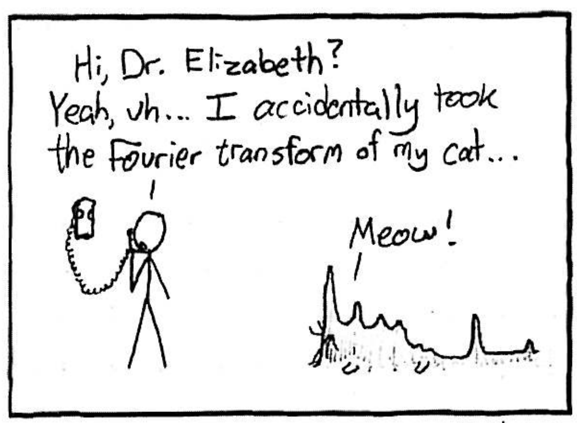
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt - a'(x)f(x,a(x)) + b'(x)f(x,b(x))$$

Fourier Transforms: $\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}u(x)dx$

$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{u}(\xi)d\xi$

| $u(x)$ | $\hat{u}(\xi)$ |
|---|---|
| $\delta(x-a)$ | $e^{ia\xi}$ |
| $e^{ikx}$ | $2\pi\delta(k+\xi)$ |
| $e^{-ax^2}$ | $\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}$ |
| $e^{-a\|x\|}$ | $\frac{2a}{a^2+\xi^2}$ |
| $H(a-\|x\|)$ | $2\frac{\sin(a\xi)}{\xi}$ |
| $u^{(n)}(x)$ | $(-i\xi)^n\hat{u}(\xi)$ |
| $u * v$ | $\hat{u}(\xi)\hat{v}(\xi)$ |

Here $H(x) = 1$ for $x \geq 0$, zero otherwise.



xkcd.com.

Greens first identity: $\int_\Omega u\Delta v + \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dA$

Product rule for divergence: $\nabla\cdot(u\mathbf{J}) = u\nabla\cdot\mathbf{J} + \mathbf{J}\cdot\nabla u$