~ SOLUTIONS ~

Math 46: Applied Math: Midterm 2

2 hours, 50 points total, 6 questions worth wildly varying numbers of points

5 1. [10 points] Consider the integral operator $Ku(x) := \int_0^1 \underbrace{xy^2}_{k(x,y)} u(y)dy$

(a) What are the eigenvalue(s) and eigenfunction(s) of this operator?

$n=1$ case (no sum needed).

Fredholm w/ Degenerate kernel $\left.\begin{array}{l}\alpha_1(x) = x \\ \beta_1(y) = y^2\end{array}\right\}$ $k(x,y) = \alpha_1(x)\beta_1(y)$

Matrix $A_{ij} = (\beta_i, \alpha_j)$ has 1 entry: $A = (x^2, x) = \int_0^1 x^3 dx = 1/4$

so matrix eigenvals are $\lambda = 1/4$ with eigenvector $\vec{c} = [1]$

Eigenvalue of K:

$\lambda_1 = 1/4$ w/ efunc $u_1(x) = \sum_{j=1}^{n} \alpha_j(x) c_j = x \cdot 1 = x$

$\lambda = 0$ (∞ multiplicity) w/ eigenspace

3 (b) Solve $Ku(x) - u(x) = x^3$ or explain why not possible.

$f(x) = x^3$

$Ku - \lambda u = f$ w/ $\lambda = 1$

$\lambda = 1$ not an eigenvalue so there is a solution for any function f:

$$\sum_{j=1}^{n} \alpha_j(x) c_j - \lambda u(x) = f(x) \quad (*) \quad \text{where } c_j := (\beta_j, u), \quad f_j := (\beta_j, f)$$

$(\beta_i, \cdot)$

$$A\vec{c} - \lambda\vec{c} = \vec{f}$$

only one component $f_1 = (x^2, x^3) = \frac{1}{6}$

$$\left(\tfrac{1}{4} - 1\right)c = \tfrac{1}{6}$$

so $c = -\frac{4}{3(6)} = -2/9$

Use $(*)$ to get $u(x) = \frac{1}{\lambda}\left(\sum \alpha_j(x) c_j - f\right)$

$= -2/9\, x - x^3$

(c) Solve $Ku(x) = x^2$, or explain why not possible. 1^{st} kind integral-equation

Only soluble if x^2 is in Range of operator K, ie in the Span $\{\alpha_j\}$ = Span$\{x\}$. This is not true $\Rightarrow$ no solution

Cheap explanation for $n=1$ case: $\forall u$, $Ku(x) = \text{const.}\, x$
$\neq x^2$!

2. [7 points] Consider the boundary-value problem $-(xu')' = f(x)$ on the interval $x \in [1, e]$ with mixed boundary conditions $u'(1) = 0$ and $u(e) = 0$.

(a) Can a Green's function exist for this problem? (Why?) Yes:

exists if 0 not an eigenvalue of $Lu := -(xu')'$ with the BCs.

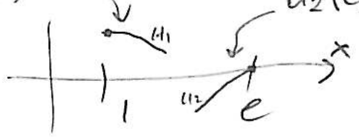
$$Lu = \lambda u = 0 \quad (\text{set } \lambda = 0) \qquad \text{so} \quad -(xu')' = 0 \overset{\text{integrate}}{\Rightarrow} xu' = c$$

$$\Rightarrow u' = \frac{c}{x} \Rightarrow \underline{u(x) = c\ln x + d} \leftarrow \text{General soln}$$

To sat. BCs, unique soln $c = d = 0$.
$\Rightarrow$ 0 not eigval.

(b) If the Green's function can exist, find it, otherwise solve the problem for general $f(x)$ another way.

$u_1'(1) = 0$ $\quad u_2(e) = 0$



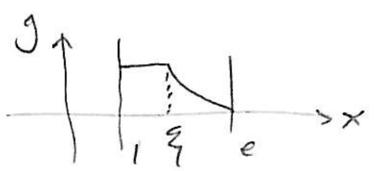
Want solutions u_1, u_2 which satisfy one BC each.

These are (up to constant factor)
$$u_1(x) = 1 \qquad (c=0,\ d=1)$$
$$u_2(x) = \ln x - 1 \qquad (c=1,\ d=-1)$$

$$W = u_1 u_2' - u_1' u_2 = 1 \cdot \frac{1}{x} - 0 \cdot (\ln x - 1) = \frac{1}{x}$$

$p(x) = x$, from the Sturm-Liouville form.

$$g(x,\xi) = \frac{-1}{p(\xi)W(\xi)} \begin{cases} u_1(x)\, u_2(\xi), & x < \xi \\ u_2(x)\, u_1(\xi), & x > \xi \end{cases} = \begin{cases} 1 - \ln\xi, & x < \xi \\ 1 - \ln x, & x > \xi \end{cases}$$



symmetric Fredholm kernel.

3. [14 points]

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(a) By converting into an ODE, find the eigenvalues and eigenfunctions of the operator $Ku(x) := \int_0^1 k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} x(1-y), & x < y \\ y(1-x), & x > y \end{cases}$$

eigenfunction relation

$$\overbrace{\lambda u = Ku}^{\text{eigenfunction relation}} = \underbrace{(1-x)\int_0^x y\, u(y)dy}_{\text{product rule}} + x\int_{x}^1 (1-y)\, u(y)dy$$

$\frac{d}{dx}$ (Leibniz.

negative sign since lower limit.

$$\lambda u' = -\int_0^x y u(y)dy + \cancel{(1-x)x u(x)} + \int_x^1 (1-y)u(y)dy \cancel{- x(1-x)u(x)}$$

$\frac{d}{dx}$

$$\lambda u''(x) = -\cancel{x} u(x) - (1-\cancel{x})u(x) = -u(x)$$

$$u'' + \frac{1}{\lambda}u = 0 \qquad \text{with BCs} \qquad \left.\begin{aligned} u(0) &= 0 \\ u(1) &= 0 \end{aligned}\right\} \text{ follow from defn of } Ku.$$

Gen. Solution is $A \sin\sqrt{\frac{1}{\lambda}}x + \cancel{B} \cos\sqrt{\frac{1}{\lambda}}x$

to match $u(0)=0$

must equal $n\pi$ when $x=1$ to match $u(1)=0$.

$$\Rightarrow \sqrt{\frac{1}{\lambda}} = n\pi \quad \text{or} \quad \underline{\lambda_n = \frac{1}{n^2\pi^2}}$$

efuncs $\underline{u_n(x) = \sin(n\pi x)}$

note $\lambda = \lambda_1$ the first eigenvalue of K.

3 (b) Solve $Ku - \boxed{\frac{1}{\pi^2}}u = \underbrace{\sin 3\pi x}_{f(x)}$, or explain why not possible.

Therefore there exists a solution only if $f(x)$ orthogonal to the λ_1 eigenspace, ie the efunc $u_1(x) = \sin \pi x$

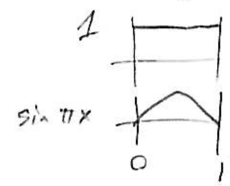
$(\sin 3\pi x, \sin \pi x) = 0$ on $[0,1]$ by Fourier sine orthogonality.

Solution is not unique: $c_i = \dfrac{f_i}{\lambda_i - \lambda}$ $f_3 = 1$ but $f_j = 0$ for $j \neq 3$.

$$\to u(x) = \underset{\text{arbitrary}}{c} \sin \pi x + \sum_{j \neq 1} \frac{f_j}{\lambda_j - \lambda} u_j(x) = c \sin \pi x + \frac{\sin 3\pi x}{\frac{1}{9\pi^2} - \frac{1}{\pi^2}}$$

(c) Solve $Ku - \frac{1}{\pi^2}u = 1$ (that is, the constant function equal to 1), or explain why not possible.

2 $f(x) = 1$ is <u>not</u> orthogonal to $u_1(x) = \sin \pi x$



$$(1, \sin \pi x) = \int_0^1 \sin \pi x \, dx = -\frac{1}{\pi}\left[\cos \pi x\right]_0^1 = -\frac{2}{\pi}$$

<u>No solution</u>.

3 (d) Solve $Ku - u = 1$, or explain why not possible.

$\lambda = 1$ not an eigenvalue so there's a solution for all f.

Need Fourier series for $f(x) = 1$:
$$f_n = \frac{(1, \phi_n)}{\|\phi_n\|^2} = \frac{\int_0^1 \sin \pi n x \, dx}{\int_0^1 \sin^2 \pi n x \, dx} \; \frac{1}{2}$$
$$= 2 \cdot \frac{-1}{n\pi}\left[\cos \pi n x\right]_0^1$$
$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

$$c_j = \frac{f_j}{\lambda_j - \lambda} = \frac{-4/j\pi}{1/n^2\pi^2 - 1} \text{ for } j \text{ odd.}$$

$$\text{So } u(x) = \sum_{j=1}^{\infty} c_j \sin \pi j x = \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{4}{j\pi\left(1 - \frac{1}{j^2\pi^2}\right)} \sin j\pi x$$

$y(0) = y(1) = 0$

4. [5 points] What can be deduced about the sign of the eigenvalues of $-y'' + xy = \lambda y$? $\rightsquigarrow$ with what BCs?

We do not know how to solve $-y'' + xy = 0$ even, directly, so can't use explicit construction.

All we have is energy method: mult. by y & integrate:

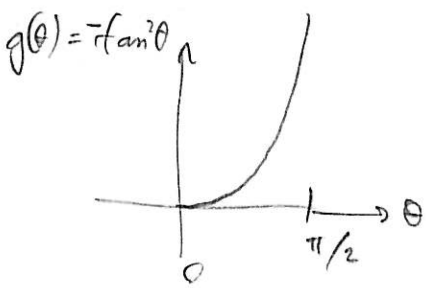
$$\underbrace{-\int yy''\,dx}_{\downarrow \text{ by parts}} + \int xy^2\,dx = \lambda \int y^2\,dx.$$

$$+\int (y')^2\,dx - \underbrace{\cancel{[yy']_0^1}}_{0}$$

so $$\lambda = \frac{\overbrace{\int_0^1 x\,y^2\,dx}^{>0} + \overbrace{\int_0^1 (y')^2\,dx}^{>0}}{\underbrace{\int_0^1 y^2\,dx}_{>0}}$$

$$\underline{> 0.}$$

5. [4 points] Find a leading-order $\lambda \gg 1$ asymptotic approximation to $\int_0^{\pi/2} e^{-\lambda \tan^2\theta}\,d\theta$



$$\int_0^{\pi/2} \underset{\uparrow\, f(\theta)=1}{} e^{\lambda g(\theta)}\,d\theta$$

with $g(\theta) = -\tan^2\theta$.

has max at $\theta = 0$ (end of interval $\Rightarrow$ 1/2 the contribution).

due to at end of interval.

so $$I \approx \boxed{\frac{1}{2}}\underbrace{f(0)}_{1}\, e^{\lambda g(0)} \sqrt{\frac{-2\pi}{\lambda g''(0)}}$$

$g(0) = 0$.

$$g'(\theta) = -2\tan\theta\,\boxed{\tan'\theta}$$

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= 1 + \tan^2\theta$$

$$I \approx \frac{1}{2}\sqrt{\frac{-2\pi}{\lambda(-2)}} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}.$$

$$g''(\theta) = -2(1+\tan^2\theta)^2 + \tan\theta(\cdots)$$

$$g''(0) = -2.$$ irrelevant

no x appears so $Ku(x)$ const w.r.t. x.

6. [10 points] Consider the operator $Ku(x) := \int_0^1 su(s)ds$. This question is a little more adventurous.

   (a) Use Cauchy-Schwarz inequality to bound the norm $||Ku||$ in terms of $||u||$, for any function u.

The const

$$Ku = (s, u(s)) = \underbrace{\int_0^1 s\, u(s)\, ds}_{Ku(x) \text{ is a const func.}} \leq \|s\| \, \|u(s)\| = \underbrace{\sqrt{\int_0^1 s^2 ds}}_{\frac{1}{\sqrt{3}}} \sqrt{\int_0^1 u^2(s) ds}$$

note since Ku = const, $\|Ku\| = \sqrt{\int (Ku(x))^2 dx} = Ku$

so $\|Ku\| \leq \frac{1}{\sqrt{3}} \|u\|$.

this states that K is a 'bounded' operator.

   (b) Even though it's a Fredholm operator you can use a Neumann series to say things about it. Write the usual Neumann series to solve the problem $u - \lambda Ku = f$. [don't be alarmed you've never had to do this before].

$$u = (1 + \lambda K + \lambda^2 K^2 + \cdots) f$$

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   (c) Leaving $f(x)$ as a general function, evaluate the first few terms of the series, simplifying as much as possible. Use this to write down an expression for the n^{th} term.

1st term $1 \cdot f(x)$

2nd term $\lambda K f(x) = \lambda \int_0^1 s f(s) ds$ — note it's a constant.

3rd term $\lambda^2 K(Kf)(x) = \lambda^2 \int_0^1 s \int_0^1 r f(r) dr \, ds$ — still a const!

$$= \lambda^2 \underbrace{\int_0^1 s\, ds}_{\frac{1}{2}} \cdot \int_0^1 r f(r) dr$$

4th term $\lambda^3 K^3 f(x) = \lambda^3 \left(\frac{1}{2}\right)^2 \int_0^1 r f(r) dr$

$\vdots$

nth term $= \lambda^{n-1} \left(\frac{1}{2}\right)^{n-2} \int_0^1 r f(r) dr = \lambda \left(\frac{\lambda}{2}\right)^{n-2} \underbrace{\int_0^1 r f(r) dr}_{\text{some const.}}$

   (d) What condition on λ makes the series converge?

Converges if ratio test < 1 : ratio $\left|\frac{n+1^{th} \text{ term}}{n^{th} \text{ term}}\right| = \left|\frac{\lambda}{2}\right| < 1$

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so $|\lambda| < 2$ gives convergence for any $f(x)$.

   (e) BONUS: By identifying when the series diverges, what do you suspect is the *spectrum* of K?

2 $|\lambda| > 2$ guarantees divergence of the series for f not orthog to x.

so $I - \lambda K$ invertible for all $|\lambda| < 2$

$\Rightarrow \mu I - K$ " " " $|\mu| > 1/2$

so |largest eigenvalue| is $1/2$! Since K is degenerate you can get its entire spectrum: $1/2$, 0 (∞ mult.)