Math 46: Applied Math: Midterm 2

2 hours, 50 points total, 6 questions worth wildly varying numbers of points

1. [10 points] Consider the integral operator $Ku(x) := \int_0^1 xy^2 u(y) dy$

   (a) What are the eigenvalue(s) and eigenfunction(s) of this operator?

   (b) Solve $Ku(x) - u(x) = x^3$, or explain why not possible.

(c) Solve $Ku(x) = x^2$, or explain why not possible.

2. [7 points] Consider the boundary-value problem $-(xu')' = f(x)$ on the interval $x \in [1, e]$ with mixed boundary conditions $u'(1) = 0$ and $u(e) = 0$.

   (a) Can a Green's function exist for this problem? (Why?)

   (b) If the Green's function can exist, find it, otherwise solve the problem for general $f(x)$ another way.

3. [14 points]

(a) By converting into an ODE, find the eigenvalues and eigenfunctions of the operator $Ku(x) := \int_0^1 k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} x(1-y), & x < y \\ y(1-x), & x > y \end{cases}$$

(b) Solve $Ku - \frac{1}{\pi^2}u = \sin 3\pi x$, or explain why not possible.

(c) Solve $Ku - \frac{1}{\pi^2}u = 1$ (that is, the constant function equal to 1), or explain why not possible.

(d) Solve $Ku - u = 1$, or explain why not possible.

4. [5 points] What can be deduced about the sign of the eigenvalues of $-y'' + xy = \lambda y$ with boundary conditions $y(0) = y(1) = 0$?

5. [4 points] Find a leading-order $\lambda \gg 1$ asymptotic approximation to $\int_0^{\pi/2} e^{-\lambda \tan^2 \theta} d\theta$

6. [10 points] Consider the operator $Ku(x) := \int_0^1 su(s)ds$. This question is a little more adventurous.

(a) Use Cauchy-Schwarz inequality to bound the norm $||Ku||$ in terms of $||u||$, for any function u.

(b) Even though it's a Fredholm operator you can use a Neumann series to say things about it. Write the usual Neumann series to solve the problem $u - \lambda Ku = f$. [don't be alarmed you've never had to do this before].

(c) Leaving $f(x)$ as a general function, evaluate the first few terms of the series, simplifying as much as possible. Use this to write down an expression for the n^{th} term.

(d) What condition on λ makes the series converge?

(e) BONUS: By identifying when the series diverges, what do you suspect is the *spectrum* of K?

Useful formulae:

Stationary phase ($c =$ interior maximum of g)

$$\int f(x)e^{\lambda g(x)}dx \approx f(c)e^{\lambda g(c)}\sqrt{\frac{-2\pi}{\lambda g''(c)}}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$