Barnett

— SOLUTIONS —

Math 46: Applied Math: Midterm 1

2 hours, 50 points total, 6 questions worth varying number of points

1. [7 points]

   In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = \text{const} \cdot l^\alpha$ when other parameters are held constant; give α.

$$\begin{array}{c|cccc} & l & E & \rho & R \\ \hline M & & 1 & 1 & 1 \\ L & 1 & 2 & -3 & -1 \\ T & & -2 & & -3 \end{array}$$

full rank $(r=3)$

so 1-dim subspace of solutions to $A\vec{x}=\vec{0}$

$\Rightarrow$ 1 dim'less param

$$\pi_1 = \frac{E^3}{l^{11}\rho R^2}$$

unlikely but true, eleven powers of l.

law must be

$$F(\pi_1) = 0 \qquad \text{ie} \qquad \pi_1 = \text{const.}$$

$$\text{so} \quad E = \text{const.}\, l^{11/3} \qquad \alpha = 11/3$$

2. [16 points. Note part c, worth 7 points, is independent of the others]

A nonlinear damped oscillator is given by the initial-value problem

$$my'' + ay' + ky^3 = 0 \qquad y(0) = 0 \qquad my'(0) = I$$

(a) If m is a mass, find the dimensions of the other three parameters a, k, I (recall y is a displacement, *i.e.* length).

$$\begin{array}{c|cccc} & m & a & k & I \\ \hline M & \multicolumn{4}{c}{\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ & & -2 & 1 \\ & -1 & -2 & -1 \end{array}\right]} \\ L & & & & \\ T & & & & \end{array}$$

(b) Write down *two* length scales and *two* time scales.

possible y_c : I/a , $\frac{a}{\sqrt{km}}$, ... in fact $I/a\left(\frac{kI^2m}{a^4}\right)^K$ ...

possible t_c : m/a , $\frac{a^3}{kI^2}$, ... or $\frac{m}{a}\left(\frac{kI^2m}{a^4}\right)^K$ $\boxed{\text{for any real } \alpha}$

(c) Show that when the model is non-dimensionalized using scaling appropriate for the *small mass* limit (choose time and length scales which don't involve m), the IVP

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1$$

results. What is ε in terms of the original parameters?

$$m\frac{y_c}{t_c^2}\bar{y}'' + a\frac{y_c}{t_c}\bar{y}' + ky_c^3\bar{y}^3 = 0 \qquad y_c\bar{y}(0) = 0$$

$$m\frac{y_c}{t_c}\bar{y}(0) = I$$

Choose $y_c = I/a \qquad t_c = \frac{a^3}{kI^2}$

divide by $\frac{kI^3}{a^3}$ so $m\frac{I}{a}\frac{k^2I^4}{a^6}\bar{y}'' + \frac{aI}{a}\frac{kI^2}{a^3}\bar{y}' + \frac{kI^3}{a^3}\bar{y}^3 = 0$

$$\Rightarrow \underbrace{\left(\frac{mI^2k}{a^4}\right)}_{\varepsilon \ll 1 \text{ if } m \text{ small}}\bar{y}'' + \bar{y}' + \bar{y}^3 = 0 \qquad \bar{y}(0) = 0$$

$$\frac{m}{a}\frac{I}{a}\frac{kI^2}{a^3}\bar{y}(0) = I$$

ie $\varepsilon\bar{y}(0) = 1$.

(d) Find a *leading-order* perturbation approximation to the solution of the IVP from (b), and give a crude sketch showing any key features. Here it is written out again:

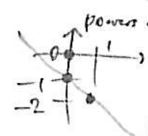
$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1 \qquad \varepsilon \ll 1$$

singular pert. $\Rightarrow$ initial layer.

inner: $\tau = \frac{t}{\delta}$ rescale.

$$\underbrace{\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y'}_{\text{balance}} + Y^3 = 0$$

so



slope $= -1$ so $\delta \varepsilon^{-1} = 1$, $\delta = \varepsilon$.

so $Y'' + Y' + \cancel{\varepsilon Y^3} = 0$ $\to 0$ leading order $\qquad Y(0) = 0 \qquad \underbrace{\frac{\varepsilon}{\delta}}_{1} Y'(0) = 1$

$$Y_i(\tau) = A e^{-\tau} + B$$

ICs fix $A + B = 0$, $-A = 1$, so $A = -1$, $B = +1$

$$Y_i(\tau) = 1 - e^{-\tau}$$

$\lim_{\tau \to \infty} Y_i(\tau) = 1 = C_m$.

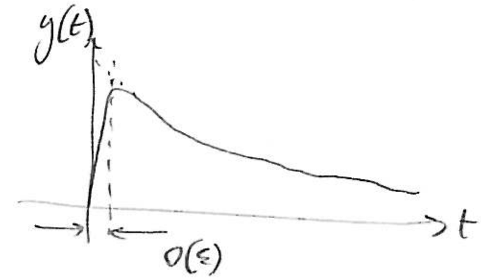
outer: ($\varepsilon = 0$)

$y_0' + y_0^3 = 0$ ie $\int \frac{dy_0}{y_0^3} = -\int dt$

$\Rightarrow \frac{1}{2} y_0^{-2} = -t + c \qquad \Rightarrow y_0(t) = \frac{1}{\sqrt{c - 2t}}$ ← c unknown.

at intermediate scale, match with inner: $C_m = \lim_{t \to 0} y_0(t) = \frac{1}{\sqrt{c - 0}} = \frac{1}{\sqrt{c}}$ so $c = 1$

$$y_a(t) = y_0 + y_i - C_m = \frac{1}{\sqrt{1 - 2t}} - e^{-t/\varepsilon}$$



ie up to $O(\varepsilon^2)$.

3. [6 points] Find a $\boxed{\text{3-term}}$ perturbation approximation to the solution of the IVP

$$y' = \frac{1}{1+\varepsilon y^2 y'} \qquad y(0) = 0$$

note this is unusual in that it's implicit for y'.

$$\frac{1}{1+x} = 1 - x + x^2 \cdots$$ by Binomial expansion

so $y' = 1 - \varepsilon y^2 y' + \varepsilon^2 y^4 y'^2 - \cdots$

Note how I only kept terms up to ε^2 th order, to keep simple.

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' \cdots = 1 - \varepsilon(y_0 + \varepsilon y_1 \cdots)^2 (y_0' + \varepsilon y_1' \cdots) + \varepsilon^2 (y_0 + \cdots)^4 (y_0' + \cdots)^2 - \cdots$$

$O(\varepsilon^0)$: $\quad y_0' = 1 \quad$ so $y_0(t) = t + c$, to match IC have $c = 0$.

$O(\varepsilon^1)$: $\quad y_1' = -y_0^2 y_0' = -t^2 \cdot 1$

so $y_1 = -\frac{1}{3}t^3 + c \quad$ with $y_1(0) = 0$ so $c = 0$.

$O(\varepsilon^2)$: $\quad y_2' = -2y_0 y_1 y_0' - y_0^2 y_1' + y_0^4 y_0'^2$

$$= +\tfrac{2}{3} t\, t^3 \cdot 1 - t^2(-t^2) + t^4 \cdot 1^2 = \tfrac{8}{3} t^4$$

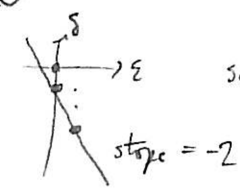
so $y_2(t) = \frac{8}{15} t^5 + c \qquad c = 0$ since ICs $y_2(0) = 0$.

put together: $\quad y(t) = t - \frac{\varepsilon}{3} t^3 + \frac{8}{15}\varepsilon^2 t^5 - \cdots$

4. [5 points] Find the leading-order perturbation approximation to all roots of $\varepsilon x^3 - x - 2 = 0$ for $\varepsilon \ll 1$.

regular roots $(\varepsilon = 0)$: $-x - 2 = 0$ $\quad x_0 = -2$

irregular roots $x = \frac{y}{\delta}$ so $\underbrace{\frac{\varepsilon}{\delta^3} y^3 - \frac{1}{\delta} y}_{\text{dom. balance}} - 2 = 0$



so $\delta^{-2}\varepsilon = 1$ or $\delta = \sqrt{\varepsilon}$

$\Rightarrow \quad y^3 - y - \cancel{\sqrt{\varepsilon}\, 2} = 0$

$\sqrt{\varepsilon}\,2 \approx 0$. leading order

so $y_0(y_0^2 - 1) = 0 \qquad y_0 = \pm 1, \ 0$

copy of regular root.

$$x = -2, \ +\frac{1}{\sqrt{\varepsilon}}, \ -\frac{1}{\sqrt{\varepsilon}}$$

5. [5 points] Find the WKB approximation for the large eigenvalues λ of

$$y'' + 4\lambda e^x y = 0 \qquad\qquad y(0) = y(1) = 0$$

$\varepsilon^2 y'' + k^2(x)\, y = 0 \qquad$ with $\quad \varepsilon^2 = \frac{1}{\lambda}$

$k(x) = \sqrt{4e^x} = 2e^{x/2}$

$$\int_0^1 k(x)dx = \left[2 \cdot 2 e^{x/2}\right]_0^1 = 4(e^{1/2} - 1)$$

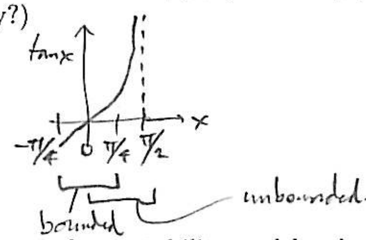
WKB soln $(x=1)$: $\sin\left(\frac{1}{\varepsilon_n}\int_0^1 k(x)\,dx\right) = 0$ to sat. BC at $x = 1$. (other at $x=0$ is ok since chose no cos WKB term)

ie $\quad \varepsilon_n \approx \frac{1}{n\pi}\int_0^1 k(x)\,dx = \frac{4(e^{1/2} - 1)}{n\pi}$

so $\quad \lambda_n \approx \frac{n^2\pi^2}{\left[\int_0^1 k(x)dx\right]^2} = \frac{n^2\pi^2}{16(e^{1/2} - 1)^2} \qquad$ for large $n \gg 1$

6. [12 points] Short answers:

(a) As $\varepsilon \to 0$ does the function $f(x,\varepsilon) = \varepsilon \tan(x)$ converge uniformly to zero on $(-\pi/4, \pi/4)$? On $(0, \pi/2)$? (Why?)



on $(-\pi/4, \pi/4)$, $|\tan x| < 1$ so uniformly to 0 as $\varepsilon \to 0$.

on $(0, \pi/2)$ tan x arb. large as $x \to \pi/2$, so not uniform.

(b) What can you say about stability and local asymptotic stability for the system $x' = -2x + y$, $y' = 4x + y$?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

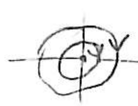
eigenvalues $(-2-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$\lambda = -3, +2$ hyperbolic (saddle)

$\Rightarrow$ neither stable nor loc. asym. stable.

(c) A linearization of a nonlinear system of two coupled ODEs at a critical points gives the Jacobean matrix $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$. What can you conclude about stability?

linear would be  clockwise center.

but in nonlinear this is the one case in which you cannot deduce stability. $\Rightarrow$ nothing known.

(d) At which end(s) would you expect the BVP $\varepsilon y'' + (1/2 - x)y' + y = 0$ with $y(0) = a$ and $y(1) = b$ to be able to support a boundary layer for $\varepsilon \ll 1$? [Do not solve the whole thing!]

both ends can have bdry layer.

near $x = 0$, $\varepsilon y'' + \frac{1}{2} y' + y \approx 0$ relative signs give decay as $\xi \to \infty \Rightarrow$ bdry layer.

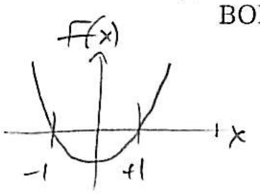
near $x = 1$ $\varepsilon y'' - \frac{1}{2} y' + y \approx 0$ so also decay as $\xi = \frac{1-x}{\varepsilon} \to \infty$.

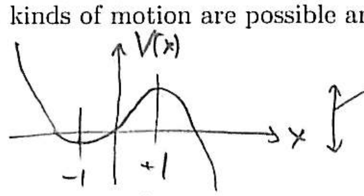
(e) State briefly in what class of problem the Poincaré-Linstedt method is needed, and what problem it fixes.

Oscillatory 2^{nd}-order ODE IVP with perturbation.

Fixes problem that period is changed by perturbation which otherwise gives secular terms which are not uniform approximation.

(f) Sketch orbits in the (x, x') plane for a particle at location $x(t)$ subjected to a force $F(x) = x^2 - 1$. BONUS: What kinds of motion are possible and in what energy range?





vertical range of size $-\int_{-1}^{1} F(x)\,dx = 4/3$ in which oscillatory.

below and above this range get motion from $x = +\infty$ returning back after a single excursion.

note overall vertical shift arbitrary (unknown).