MATH 46 WORKSHEET: Eigenmodes

5/23/07
Barnett

Which of the following problems are <u>separable</u>? (don't solve, just find BCs on $X(x), Y(y)$, etc).

i) $-\Delta u = \lambda u$ in $\Omega = [0,1] \times [0,2]$ with
$$u(x,0) = u(x,2) = 0, \quad 0<x<1$$
$$u_x(0,y) = u_x(1,y) = 0, \quad 0<y<2$$

ii) $-\Delta u = \lambda u$ in $\Omega = [0,1] \times [0,2]$ with
$$u(x,0) = 0 \qquad 0<x<1$$
$$u_y(x,2) = 0 \qquad 0<x<1$$
$$u(0,y) = 0 \qquad 0<y<2$$
$$u_x(1,y) = 0 \qquad 0<y<2$$

Find eigenfunctions of $-\Delta u = \lambda u$ in the unit disc with zero Dirichlet BCs using separation of variables and $\Delta u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$. [Solve for $\Theta(\theta)$ modes first, then leave ODE for $R(r)$]

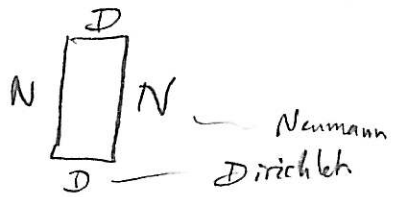
MATH 46 WORKSHEET : Eigenmodes

5/23/07
Barnett

Which of the following problems are <u>separable</u>? (don't solve, just find BCs on $X(x), Y(y)$, etc).

i) $-\Delta u = \lambda u$ in $\Omega = [0,1] \times [0,2]$ with $u(x,0) = u(x,2) = 0$, $0<x<1$

$u_x(0,y) = u_x(1,y) = 0$, $0<y<2$



✓

$X'(0) = X'(1) = 0$

$Y(0) = Y(2) = 0$

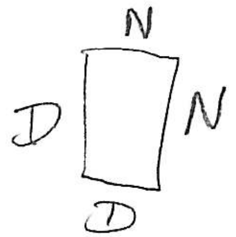
ii) $-\Delta u = \lambda u$ in $\Omega = [0,1] \times [0,2]$ with

$u(x,0) = 0 \quad 0<x<1$

$u_y(x,2) = 0 \quad 0<x<1$

$u(0,y) = 0 \quad 0<y<2$

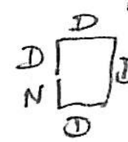
$u_x(1,y) = 0 \quad 0<y<2$



✓ also separable!

$X(0) = 0 \quad X'(1) = 0$

$Y(0) = 0 \quad Y'(2) = 0$

Note


would not be.

Find eigenfunctions of $-\Delta u = \lambda u$ in the unit disc with zero Dirichlet BCs using separation of variables and $\Delta u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$. [Solve for $\Theta(\theta)$ modes first, then leave ODE for $R(r)$]

$u(r,\theta) = R(r)T(\theta)$

Eigenvalue relation:

$$\text{so } -\Delta u = -\frac{1}{r}(rR')'T - \frac{1}{r^2}RT'' = \lambda RT = \lambda u$$

Divide by RT: $$-\frac{1}{rR}(rR')' - \frac{1}{r^2}\frac{T''}{T} = \lambda$$

Mult. by r^2 to split up r from t terms: $$\underbrace{-\frac{r}{R}(rR')' - \lambda r^2}\; - \underbrace{\frac{T''}{T}}_{= \text{const} = \mu} = 0$$

so $r(rR')' - \lambda r^2 R = \mu = n^2$

$\Rightarrow \quad r^2R'' + rR' + (n^2 - \lambda r^2)R = 0$

with $R(0)$ bounded & $R(1) = 0$

This is Bessel's Equation, solutions are Bessel Funcs. (see Boyce & DiPrima).

$= \text{const} = \mu$ so $T'' + \mu T = 0$ with $T(0) = T(2\pi)$ periodic BCs.

$\Rightarrow \mu = n^2$ for $n \in \mathbb{Z}$.

eigenfuncs $T_n(\theta) = e^{in\theta}$ or $\cos k$, $\sin$