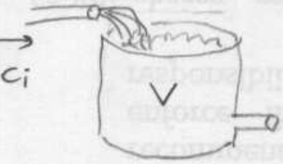
MATH 46 WORKSHEET: Scaling & Non-dimensionalizing F 3/30/07

Consider a chemical reactor tank with flow rate q, volume V, incoming concentration of reactant c_i. We stir the tank so concentration inside, $c(t)$, is uniform, so (chemical) mass inside is $Vc(t)$.



Inside the tank the reactant decays at rate k, ie rate of loss of mass is $k\boxed{Vc(t)}$ (mass inside)

Write an ODE expressing mass balance: $\frac{d}{dt}(Vc(t)) = \underline{\qquad}$ (mass arrival rate) $- \underline{\qquad}$ (loss rate)

divide it by V:

Initial Condition is $c(0) = c_0$

How many params in model?

Rewrite ODE & IC using general nondimensionalized $\tau = \frac{t}{t_c}$, $\bar{c} = \frac{c}{c_c}$

) Choose $t_c = k^{-1}$ and $c_c = c_i$, rewrite ODE & IC, expressing in terms of $\gamma = \frac{c_i}{c_0}$ and $\beta = \frac{kV}{q}$ (dimensionless params)

) Instead change t_c to the other timescale derivable from original problem params. Keep c_c as before & rewrite ODE & IC

) if in reality $\beta \ll 1$ (what is the interpretation?) which of A or B is appropriate?

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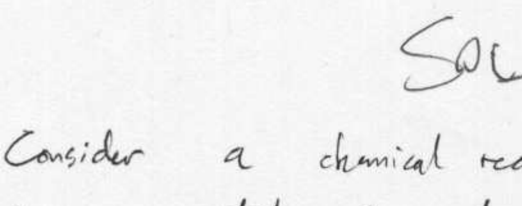
SOLUTIONS.

Consider a chemical reactor tank with flow rate q, volume V, incoming concentration of reactant c_i. We stir the tank so concentration inside, $c(t)$,



is uniform, so (chemical) mass inside is $Vc(t)$.

Inside the tank the reactant decays at rate k, ie rate of loss of mass is $k\boxed{Vc(t)}$ mass inside

Write an ODE expressing mass balance : $\frac{d}{dt}(Vc(t)) = \overbrace{qc_i}^{\text{mass arrival rate}} - \overbrace{(qc + kVc)}^{\text{loss rate}}$

divide it by V: $\dot{c} = \frac{q}{V}(c_i - c) - kc$

Initial Condition is $c(0) = c_0$

How many params in model? 5: $q \quad V \quad c_i \quad c_0 \quad k$.

Rewrite ODE & IC using general nondimensionalized $\bar{t} = \frac{t}{t_c}$, $\bar{c} = \frac{c}{c_c}$

DE
$$\frac{c_c}{t_c}\frac{d\bar{c}}{d\bar{t}} = \frac{qc_c}{V}\left(\frac{c_i}{c_c} - \bar{c}\right) - kc_c\bar{c}$$

IC
$$c_c\bar{c}(0) = c_0$$

Choose $t_c = k^{-1}$ and $c_c = c_i$, rewrite ODE & IC, expressing in terms of $\gamma := \frac{c_i}{c_0}$ and $\beta := \frac{kV}{q}$ (dimensionless params).

$$\cancel{k}\cancel{c_i}\frac{d\bar{c}}{d\bar{t}} = \cancel{k}\underbrace{\left(\frac{q}{kV}\right)}_{\frac{1}{\beta}}\cancel{c_i}(1-\bar{c}) - \cancel{k}\cancel{c_i}\bar{c}$$ ie cancel c_i and divide by k

ie $\begin{cases} \frac{d\bar{c}}{d\bar{t}} = \frac{1}{\beta}(1-\bar{c}) - \bar{c} \\ \bar{c}(0) = 1/\gamma \end{cases}$

) Instead change t_c to the other timescale ($t_c = \frac{V}{q}$) derivable from original problem params. Keep c_c as before & rewrite ODE & IC

$$\frac{\cancel{c_i}q}{V}\frac{d\bar{c}}{d\bar{t}} = \frac{q}{V}\cancel{c_i}(1-\bar{c}) - k\cancel{c_i}\bar{c}$$ ie $\begin{cases} \frac{d\bar{c}}{d\bar{t}} = 1 - \bar{c} - \beta\bar{c} \\ \bar{c}(0) = 1/\gamma \end{cases}$

) if in reality $\beta \ll 1$ (what is the interpretation? decays much slower than it flows.) which of A or B is appropriate? since A gives garbage as $\beta \to 0$.