Trigonometric Integrals

Basic Technique

Suppose that $U(x, y)$ is a continuous function on $R = \{ (x, y) \in \mathbf{R}^2 : -1 \le x, y \le 1 \}$. Then

$$\int_0^{2\pi} U\big(\cos(\theta), \sin(\theta)\big)\, d\theta = \int_{|z|=1} F(z)\, dz = 2\pi i \sum_{|z|<1} \operatorname{Res}(F; z)$$

where

$$F(z) := U\left(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right) \cdot \frac{1}{iz}.$$

Improper Adventures

- The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges only when both $\int_{a}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ converge for some (hence all) $a \in (-\infty, \infty)$.
- We define p.v. $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$.
- If $\int_{-\infty}^{\infty} f(x)\,dx$ converges, then $\int_{-\infty}^{\infty} f(x)\,dx = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx$. But the converse can fail.

Back to Calculus II

Theorem (Comparison Theorem)

Suppose that $|f(x)| \leq g(x)$ *for all* $x \in \mathbf{R}$. *Then if* $\int_{-\infty}^{\infty} g(x)\,dx$ *converges, so does* $\int_{-\infty}^{\infty} f(x)\,dx$.

Theorem (Plus Two)

Suppose that $p(x)$ *and* $q(x)$ *are polynomials with real coefficients such that*

$$\deg p(x) + 2 \leq \deg q(x)$$

and that $q(x)$ *has no zeros on the real axis. Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$$

converges.

Limits

Lemma (Crude Limit Lemma)

Suppose that $p(z)$ *and* $q(z)$ *are polynomials with* $\deg p(z) + 2 \leq \deg q(z)$. *Let*

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \text{with } a \geq 0.$$

Let C_R^+ *be the top half of the positively oriented circle* $|z| = R$ *from* R *to* $-R$. *Then*

$$\lim_{R \to \infty} \int_{C_R^+} F(z)\, dz = 0.$$

Remark

Note that a must be real and that the special case $a = 0$ applies to all rational functions with the degree of the denominator at least 2 more that the degree of the denominator.

Today

Theorem (Improper Integrals Plus 2)

Suppose that $p(z)$ and $q(z)$ are polynomials such that

$$\deg p(z) + 2 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \textit{with } a \geq 0.$$

If $q(z)$ has no zeros on the real-axis, then

$$\int_{-\infty}^{\infty} F(x)\, dx = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z).$$