CIF for the Derivatives

Theorem (Cauchy's Integral Formula for the Derivatives)

Suppose that f is analytic on and inside a simple closed contour Γ. Let D be the interior of Γ. Then for all $n \geq 0$, $f^{(n)}(z)$ exists for all $z \in D$ and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z)^{n+1}}\, d\omega. \quad \text{for all } z \in D.$$

Remark

Note that the case $n = 0$ is just the usual Cauchy Integral Formula.

Good Times

Theorem (Morera's Theorem)

Suppose that f is continuous on a domain D and that for all closed contours Γ in D we have

$$\int_\Gamma f(z)\, dz = 0.$$

The f is analytic on D.

Theorem (Cauchy's Estimates)

Suppose that f is analytic on $B_R(z_0)$ and that $|f(z)| \leq M$ for all $z \in B_R(z_0)$. Then for $n = 0, 1, 2, \ldots,$ we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

More Good Stuff

Theorem (Liouville's Theorem)

A bounded entire function must be constant.

Theorem (Maximum Modulus Principle)

Suppose that f is analytic in a domain D and that there is a $z_0 \in D$ such that

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in D.$$

Then f is constant.

Remark (Bounded Regions)

- Recall that a domain D is bounded if there is a $R > 0$ such that $D \subset B_R(0)$.
- The boundary ∂D of D is the set of points z such that every open ball $B_r(z)$ contains points in D and not in D.
- The closure $\overline{D}$ of D is the union of D and ∂D. Of course, $\overline{D}$ is closed.
- If D is bounded, then $\overline{D}$ is closed and bounded.
- Thus if D is a bounded domain, then any continuous real-valued function on $\overline{D}$ must attain its maximum and minimum on $\overline{D}$.