Limits

The key to our "standard approach" to improper integrals is our "Crude Limit Lemma":

Lemma (Crude Limit Lemma)

Suppose that $p(z)$ and $q(z)$ are polynomials with $\deg p(z) + 2 \leq \deg q(z)$. Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \text{with } a \geq 0.$$

Let C_R^+ be the top half of the positively oriented circle $|z| = R$ from R to $-R$. Then

$$\lim_{R\to\infty} \int_{C_R^+} F(z)\, dz = 0.$$

This is the essential ingredient in the proof of our basic result from last lecture.

Theorem (Improper Integrals Plus 2)

Suppose that $p(z)$ and $q(z)$ are polynomials such that

$$\deg p(z) + 2 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \text{with } a \geq 0.$$

If $q(z)$ has no zeros on the real-axis, then

$$\int_{-\infty}^{\infty} F(x)\,dx = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z).$$

Proof

Proof of Theorem II+2.

For large $R > 0$, all the singularities of F lie inside $\Gamma_R = [-R, R] + C_R^+$. Hence

$$2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z) = \int_{\Gamma_R} F(z)\, dz = \int_{-R}^{R} F(z)\, dx + \int_{C_R^+} F(z)\, dz.$$

Now we let $R \to \infty$ and use our CL Lemma and what we know about convergence of improper integrals. $\square$

Improper Integrals with Trigonometric Functions

Theorem (Improper Integrals Plus 1)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that

$$\deg p(z) + 1 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

Then if $q(x)$ has no zeros on the real axis and $a > 0$ then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big(2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z)\Big)$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big(2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}(F; z)\Big)$$

The Textbook's Approach

Remark

The previous theorem does not apply to integrals such as

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 4i}\, dx.$$

The denominator is not a polynomial with real coefficients so

$$\mathrm{Re}\Big(\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 4i}\, dx\Big) \neq \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 4i}\, dx.$$

The approach in the textbook can deal with this situation, but involves computing all the residues of F—not just those in the upper half-plane—and hence considerably more algebra. We are not going to worry about such things in this course.