Last Time

Recall that or $z = |z|e^{i\theta} \neq 0$, we have

$$\begin{aligned} \log(z) &= \{\, w \in \mathbb{C} : e^w = z \,\} \\ &= \ln(|z|) + i \arg(z) \\ &= \{\, \ln(|z|) + iy : y \in \arg(z) \,\}. \end{aligned}$$

Definition

If $f(z)$ is a multiple-valued function in a domain D, then we say that a continuous function F on D is a branch of $f(z)$ in D if $F(z) \in f(z)$ for all $z \in D$.

Examples

Lemma

If $\mathcal{L}_\tau(x) = \ln(|z|) + i \arg_\tau(z)$*, then* $\mathcal{L}_\tau$ *is an analytic branch of* $\log(z)$ *in* D^*_τ *with* $\frac{d}{dz}(\mathcal{L}_\tau(z)) = \frac{1}{z}$*.*

Definition

We call $\mathrm{Log}(z) := \mathcal{L}_{-\pi}(z)$ the principal branch of $\log(z)$.

Note that $\mathrm{Log}(z)$ is an analytic branch of $\log z$ in $D^* = D^*_{-\pi}$.

Remark

The question of whether or not there is an analytic branch of a multiple-valued function $f(z)$ such as $\log z$ in a given domain D is very subtle. We saw by example, that $\log z$ can have an analytic branch in a domain D by cleverly pasting together multiple $\mathcal{L}_\tau$'s.