Fundamental Theorem for Contour Integrals

Theorem (Fundamental Theorem for Contour Integrals)

Suppose that f is continuous on a domain D and that F is an antiderivative for f in D. (That is, F is analytic in D and $F'(z) = f(z)$ for all $z \in D$.) Let Γ be a contour in D with initial point ω_0 and terminal point ω_1. Then

$$\int_\Gamma f(z)\, dz = F(\omega_1) - F(\omega_0).$$

Antiderivatives

Theorem (Antiderivatives)

Suppose that f is continuous on a domain D. Then the following are equivalent.

1. *f has an antiderivative on D.*
2. *For every closed contour Γ in D we have*

$$\int_{\Gamma} f(z)\,dz = 0.$$

3. *If Γ_0 and Γ_1 are contours in D with the same initial and terminal points, then*

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

In other words, contour integrals of f in D are path independent.

Deformations or Homotopies

Definition

Suppose that Γ_0 and Γ_1 are closed contours in a domain D. Then Γ_0 can be continuously deformed into Γ_1 in D if there is a continuous function $z : [0,1] \times [0,1] \to D$ such that

1. for each $s \in (0,1)$, $t \mapsto z(s,t)$ is an admissible parameterization of a closed contour Γ_s in D,
2. $t \mapsto z(0,t)$ is an admissible parameterization of Γ_0, and
3. $t \mapsto z(1,t)$ is an admissible parameterization of Γ_1.

Note that for any $s \in [0,1]$, we allow for Γ_s to be a point with parameterization given by the constant function.

Remark

In the text, the authors call a closed contour a loop. In many texts, one says Γ_0 and Γ_1 are homotopic if one can be continuously deformed into the other.