1. (9) Complete the following ***definitions***.

   (a) A subset D of the complex plane is called a *domain* if ...

   ANS: D is both open and connected.

   (b) We say that f is *analytic* at a point z_0 is a domain D if ...

   ANS: If $f'(z)$ exists for all z in a neighborhood of z_0.

   (c) We say that v is a *harmonic conjugate* to a harmonic function u in a domain D if ...

   ANS: v is harmonic in D, and the function $f(x+iy) = u(x,y) + iv(x,y)$ is analytic in D.

2. (8) Suppose that D is a domain and that $f : D \to \mathbf{C}$ is given by

$$f(x+iy) = u(x,y) + iv(x,y)$$

for real-valued functions u and v. Let $z_0 = a + ib$.

   (a) What are the Cauchy-Riemann equations for f at z_0?

   ANS:

$$\begin{aligned} u_x(a,b) &= v_y(a,b) \\ u_y(a,b) &= -v_x(a,b). \end{aligned}$$

   (b) How are the Cauchy-Riemann equations related to the differentiability of f at z_0? (Discuss both necessary and sufficient conditions.)

   ANS: If f is differentiable at z_0, then u and v have partial derivatives at z_0 and the Cauchy-Riemann equations must hold at z_0.

   The converse does not hold without additional assumptions. What we proved in class is that if the partial derivatives of u and v exist in a neighborhood of z_0, if they are continuous at z_0, and if the Cauchy-Riemann equations hold at z_0, then f is differentiable at z_0.

   Comment: Since I more or less told the class that this question would be on the exam, I was fairly picky. You had to have separate necessary and sufficient conditions and you had to all three criteria for sufficiency clearly stated.

3. (5) Suppose that $f(z) = e^{-iz}$. Find formulas for u and v so that

$$f(x+iy) = u(x,y) + iv(x,y),$$

and u and v are real-valued.

ANS: We have $f(x+iy) = e^{-i(x+iy)} = e^{y-ix} = e^{y}e^{-ix} = e^{-y}\big(\cos(-x) + i\sin(-x)\big)$. Thus $\boxed{u(x,y) = e^{y}\cos(x) \text{ and } v(x,y) = -e^{y}\sin(x)}$.

Comment: We're not in junior high school. Mathematicians do not present final answers with expressions like $\cos(-x)$ in them.

$u(x,y) =$ ______________________________

$v(x,y) =$ ______________________________

4. (5) Evaluate $(\sqrt{3}+i)^9$. (Your answer should be in the form $a+ib$.)

ANS: If $z = \sqrt{3}+i$, then $r = 2$ and an argument is clearly $\pi/6$. Then

$$(\sqrt{3}+i)^9 = (2e^{i\frac{\pi}{6}})^9 = 2^9 e^{i\frac{3}{2}} = 512(-i) = \boxed{-512i}.$$

$(\sqrt{3}+i)^9 =$ ______________________________

5. (6) Factor $1+z+z^2+z^3+z^4+z^5$ as a product of linear factors and irreducible quadratics all with real coefficients.

ANS: We have $p(z) = 1+z+z^2+z^3+z^4+z^5 = \frac{z^6-1}{z-1}$. Hence the roots of $p(z)$ are the sixth roots of 1 (with the exception of $z = 1$ which is clearly not a root). The sixth roots of 1 are -1 together with ω_1 and $\overline{\omega_1}$ where $\omega_1 = e^{i\frac{\pi}{3}}$ and ω_2 and $\overline{\omega_2}$ where $\omega_2 = e^{i\frac{2\pi}{3}}$. Thus,

$$\begin{aligned} p(z) &= (z+1)(z-\omega_1)(z-\overline{\omega_1})(z-\omega_2)(z-\overline{\omega_2}) \\ &= (z+1)(z^2 - 2(\operatorname{Re}\omega_1)z + 1)(z^2 - 2(\operatorname{Re}\omega_2) + 1) \\ &= (z+1)(z^2 - z + 1)(z^2 + z + 1). \end{aligned}$$

$1+z+z^2+z^3+z^4+z^5 =$ ______________________________

6. (12) For each statement, circle either **T** for "True" or **F** for "False". You ***do not have to justify your answer***.

ANS: Hey: Ten questions, twelve points. In honor of Woody Allen, you get two points for just showing up. (Apparently, that is all the honor Woody deserves.)

T **F** (a) If v is a harmonic conjugate of u in a domain D, then u is a harmonic conjugate of v in D.

ANS: FALSE: consider $u = x^2 - y^2$ and $v = 2xy$.

T **F** (b) For all $z \in \mathbf{C}$, we have $\operatorname{Log}(e^z) = z$.

ANS: FALSE: Consider $z = 2\pi i$.

T **F** (c) If u and v are Harmonic in a domain D, then so is uv.

ANS: FALSE: Consider $u = v = x$.

T **F** (d) If u and v are harmonic in a domain D, then $f(x + iy) = u(x, y) + iv(x, y)$ is analytic in D.

ANS: FALSE: Consider $u(x, y) = x$ and $v(x, y) = -y$.

T **F** (e) For all z in the domain of Log, $e^{\operatorname{Log}(z)} = z$.

ANS: TRUE: Any element w in $\log z$ satisfies $e^w = z$.

T **F** (f) For all $z \in \mathbf{C}$, we have $|\cos(z)| \le 1$.

ANS: FALSE: Consider $z = iy$ with y real.

T **F** (g) We have $\lim_{z\to\infty} |e^z| = \infty$.

ANS: FALSE: Consider $z = iy$ as above.

T **F** (h) If u and v are real valued (with continuous second partial derivatives) and if $f(x + iy) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are Harmonic in D.

ANS: This was a theorem proved in class.

T **F** (i) For all $z, w \in \mathbf{C} \setminus \{ 0 \}$, we have $\log(zw) = \log(z) + \log(w)$.

ANS: TRUE: We proved this in lecture.

T **F** (j) For all $z, w \in \mathbf{C}$, we have $e^{z+w} = e^z e^w$.

ANS: TRUE: We proved this in lecture.

Math 43 — Exam I — Take Home Portion

Problems #1–#6 are to be completed in class on Thursday. The remaining problems are to be turned in at the beginning of class on Friday (April 24^{th}). Your solutions are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. The in-class portion of the exam is "closed book". For the take-home portion, you may consult the text and your class notes, but no other sources are allowed. In both cases, you are to work alone and neither give nor receive help from anyone excepting only that you may ask me for clarification.

7. (6) Find all complex $z = a + ib$ such that $(1 + iz)^6 = 64z^6$. (Note that your answers must be in the form $a + ib$ with a and b real — and $64 = 2^6$!)

ANS: Full disclosure: the arithmetic here was much more involved than I intended. That was a mistake on my part. I'm sorry.

Anyway, we want

$$\Bigl(\frac{1+iz}{z}\Bigr)^6 = 2^6.$$

The sixth roots of 2^6 are $\omega_k = 2\exp\Bigl(i\frac{2\pi k}{6}\Bigr)$ for $k = 0, 1, 2, 3, 4, 5$. That is: $2, 2e^{i\frac{\pi}{3}}, 2e^{i\frac{2\pi}{3}}, -2, 2e^{i\frac{4\pi}{3}}, 2e^{i\frac{5\pi}{3}}$ or

$$2, \quad 1 + i\sqrt{3}, \quad -1 + i\sqrt{3}, \quad -2, \quad 1 - i\sqrt{3}, \quad -1 - i\sqrt{3}.$$

Now we want $\dfrac{1+iz}{z} = \omega_k$ or

$$\begin{aligned} z &= \frac{1}{\omega_k - i} \\ &= \frac{\overline{\omega_k} + i}{(\omega_k - i)(\overline{\omega_k} + i)} \\ &= \frac{\overline{\omega_k} + i}{|\omega_k|^2 + i(\omega_k - \overline{\omega_k}) + 1} \\ &= \frac{\overline{\omega_k} + i}{5 - 2\,\mathrm{Im}(\omega_k)}. \end{aligned}$$

Hence we get solutions (in order):

$$\frac{2+i}{5}, \quad \frac{1 + i(1 - \sqrt{3})}{5 - 2\sqrt{3}}, \quad \frac{-1 + i(1 - \sqrt{3})}{5 - 2\sqrt{3}}, \quad \frac{-2+i}{5}, \quad \frac{1 + i(1 + \sqrt{3})}{5 + 2\sqrt{3}}, \quad \frac{-1 + i(1 + \sqrt{3}}{5 + 2\sqrt{3})}.$$

8. (8) Let ω be the 5^{th} root of unity $e^{\frac{2\pi i}{5}}$.

(a) Explain why $1 + \omega + \overline{\omega} + \omega^2 + \overline{\omega^2} = 0$. (Hint: consider $1 + \omega + \omega^2 + \omega^3 + \omega^4$.)

ANS: The usual geometric sum gives $1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{\omega^5 - 1}{\omega - 1} = \frac{0}{\omega - 1} = 0$. But $\omega^4 = \overline{\omega}$ and $\omega^3 = \overline{\omega^2}$. Hence

$$1 + \omega + \overline{\omega} + \omega^2 + \overline{\omega^2} = 1 + \omega + \omega^4 + \omega^2 + \omega^3 = 0$$

as claimed.

(b) Use part (a) to find a nice formula for $\cos\left(\frac{2\pi}{5}\right)$. (For example, $\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$.)

ANS: Let $\omega = a + ib$ so that $a = \cos\left(\frac{2\pi}{5}\right)$ and $b = \sin\left(\frac{2\pi}{5}\right)$. Then $2a = \omega + \overline{\omega}$. On the other hand, $\omega^2 + \overline{\omega^2} = 2\operatorname{Re}(\omega^2) = 2(a^2 - b^2) = 2(2a^2 - 1)$ (since $a^2 + b^2 = 1$).

Plugging into the sum from part (a) we get

$$1 + (2a) + (4a^2 - 2) = 0 = 4a^2 + 2a - 1.$$

By the quadratic formula,

$$a = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-1 \pm \sqrt{5}}{4}.$$

Since we know $a > 0$, we get

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}.$$

9. (8) Recall that

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

Let

$$u(x, y) := \cos(x)\cosh(y).$$

Is there a function v such that $f(x + iy) = u(x, y) + iv(x, y)$ is entire? If so, find v, and give a formula for $f'(x + iy)$.

ANS: It is not hard to check that u is harmonic. (However, this is not strictly necessary for the problem, since if we find v such that f is analytic, then u must be harmonic as the real part of an analytic function.) Anyway, we are asked if we can find a harmonic conjugate for u. For this, we must have

$$v_y(x, y) = u_x(x, y) = -\sin(x)\cosh(y).$$

Hence $v(x, y) = -\sin(x)\sinh(y) + C(x)$ for some function C. On the other hand, we must also have

$$-\cos(x)\sinh(y) + C'(x) = v_x(x, y) = -u_y(x, y) = -\cos(x)\sinh(y).$$

Therefore we need $C'(x) = 0$, and $C(x) = c$ for some constant $c \in \mathbf{R}$. Therefore we can let

$$v(x, y) = -\sin(x)\sinh(y) + c$$

for any constant c. Then u and v satisfy the Cauchy-Riemann equations everywhere and their partials are continuous. Thus,

$$f(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) + ic$$

is an entire function. As we learned in lecture, the derivative is given by

$$f'(x + iy) = u_x(x, y) + iv_x(x, y) = -\sin(x)\cosh(y) - i\cos(x)\sinh(y).$$

COMMENT: Taking $c = 0$, we have $f(z) = \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$. Some of you used this to work the problem. This is not what I had in mind, but it is nevertheless clearly correct.

10. (8) Let $z^{\frac{1}{3}} = \{\, \omega \in \mathbf{C} \setminus \{0\} : \omega^3 = z \,\} = \exp\bigl(\frac{1}{3}\log(z)\bigr)$.

(a) Suppose that f and g are analytic branches of $z^{\frac{1}{3}}$ in a domain $D \subset \mathbf{C} \setminus \{0\}$. Show that for all $z \in D$ we have $f(z) = \omega g(z)$ where ω is a cube root of 1. (Hint: note that neither g nor f can be zero in D. Hence $h = f/g$ is also a well-defined nonzero function on D.)

ANS: Since g and f are never zero in D, $h = f/g$ is nonzero and analytic in D. Furthermore $h^3(z) = 1$ for all $z \in D$. Hence $3h(z)^2h'(z) = 0$. Since h is nonzero, it follows that $h'(z) = 0$ for all $z \in D$. Therefore h is a constant. But for any fixed $z_0 \in D$, we must have $f(z_0) = \omega g(z_0)$ (since $f(z_0)/g(z_0)$ is a cube root of 1). Hence $h(z_0) = \omega$. Since h is constant, $f(z) = \omega g(z)$ for all $z \in D$.

Comment: This comment applies to both part (a) and (b). As I reminded the class on Wednesday, unless you provide a proof otherwise, a branch f of $z^{\frac{1}{n}}$ does not have to have anything at all to do with a branch of $\log z$. Thus you can't even assume $f(z) = \exp(\frac{1}{n}g(z))$ where g is a branch of $\log z$.

(b) Show that there is no analytic branch of $z^{\frac{1}{3}}$ defined in $\mathbf{C} \setminus \{0\}$. (Hint: you may assume that $r(z) = \exp\bigl(\frac{1}{3}\operatorname{Log}(z)\bigr)$ has a jump discontinuity along the negative real axis.)

ANS: Suppose that f is such a branch. Let $r(z) = \exp(\frac{1}{3}\operatorname{Log}(z))$. Then f and r define branchs of $z^{\frac{1}{3}}$ in D^*. Hence, by part (a), for $z \in D^*$, $f(z) = \omega r(z)$. But r has a jump discontinuity along the negative real axis. Hence, so does f. Thus such an f would not even be continuous on the negative real axis. This is a contradiction, so f can't exist.

NAME : ______________________________

Math 43

23 April 2015
Dana P. Williams

Problems #1–#6 are to be completed in class on Thursday. The remaining problems are to be turned in at the beginning of class on Friday (April 24^{th}). Your solutions are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. The in-class portion of the exam is "closed book". For the take-home portion, you may consult the text and your class notes, but no other sources are allowed. In both cases, you are to work alone and neither give nor receive help from anyone excepting only that you may ask me for clarification.

Problem	Points	Score
1	9	
2	8	
3	5	
4	5	
5	6	
6	12	
7	6	
8	8	
9	8	
10	8	
Total	75	