

Worksheet – Functions – Math 3 – Jan 4, 2012

One way to build functions is by composition, i.e. plugging one function into another. If $f(x)$ and $g(x)$ are functions, then for whatever x for which $g(x)$ is in the domain of $f(x)$, then we can write

$$(f \circ g)(x) = f(g(x)).$$

For example, if $f(x) = \frac{x+1}{3x-2}$ and $g(x) = \sqrt{x}$, then

$$(f \circ g)(x) = \frac{\sqrt{x}+1}{3\sqrt{x}-2} \quad (\text{for } x \neq \pm(2/3)^2)$$

and

$$(g \circ f)(x) = \sqrt{\frac{x+1}{3x-2}} \quad (\text{whenever } \frac{x+1}{3x-2} \geq 0).$$

1. Let $f(x) = \frac{x+1}{3x-2}$ and $g(x) = \frac{1}{x}$.

(a) Calculate $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$f \circ g(x) = \frac{\frac{1}{x} + 1}{3(\frac{1}{x}) - 2}$$

$$g \circ f(x) = \frac{1}{\left(\frac{x+1}{3x-2}\right)} = \frac{3x-2}{x+1} \quad \text{for all } x \neq 2/3$$

(b) What is the domain of $(g \circ f)(x)$?

[hint: Careful! The domain of $(g \circ f)(x)$ is the set of x 's which satisfy both (1) $f(x)$ exists, and (2) $(g \circ f)(x)$ exists.]

$$(-\infty, -1) \cup (-1, 2/3) \cup (2/3, \infty)$$

since the domain of f excludes $2/3$,
and the domain of $\frac{3x-2}{x+1}$ excludes

$$x = -1.$$

2. Let $f = \frac{x+1}{3x-2}$

(a) Calculate $f^{-1}(x)$.

Set $x = f(y) = \frac{y+1}{3y-2}$ and

solve for y :

$$x(3y-2) = y+1$$

$$3xy - 2x = y + 1$$

So $3xy - y = 2x + 1$

$$y(3x-1) =$$

(collect y terms
on one side, and
non y terms on
the other)

So $f^{-1}(x) = y = \frac{2x+1}{3x-1}$

(b) Check your answer to #1 by explicitly calculating $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$ (you should get x both times).

$$f \circ f^{-1}(x) = \frac{\left(\frac{2x+1}{3x-1}\right) + 1}{3\left(\frac{2x+1}{3x-1}\right) - 2} = \frac{\left(\frac{2x+1+3x-1}{3x-1}\right)}{\left(\frac{6x+3-6x+2}{3x-1}\right)}$$

↑
common denominators

$$= \frac{\left(\frac{5x+0}{3x-1}\right)}{\left(\frac{5}{3x-1}\right)} = \frac{5x}{5} = x \checkmark$$

$$f^{-1} \circ f(x) = \frac{2\left(\frac{x+1}{3x-2}\right) + 1}{3\left(\frac{x+1}{3x-2}\right) - 1} = \frac{\left(\frac{2x+2+3x-2}{3x-2}\right)}{\left(\frac{3x+3-3x+2}{3x-2}\right)}$$

$$= \frac{5x}{5} = x \checkmark$$

- (c) If $(f \circ g)(x) = x + 2$, what is $g(x)$?
[hint: since $(f \circ g)(x) = f(g(x)) = x + 2$, we know

$$g(x) = f^{-1}(f(g(x))) = f^{-1}(x + 2).]$$

$$f(g(x)) = x + 2$$

$$\text{so } g(x) = f^{-1}(x + 2)$$

$$= \frac{2(x + 2) + 1}{3(x + 2) - 1}$$

$$= \boxed{\frac{2x + 5}{3x + 5}}$$